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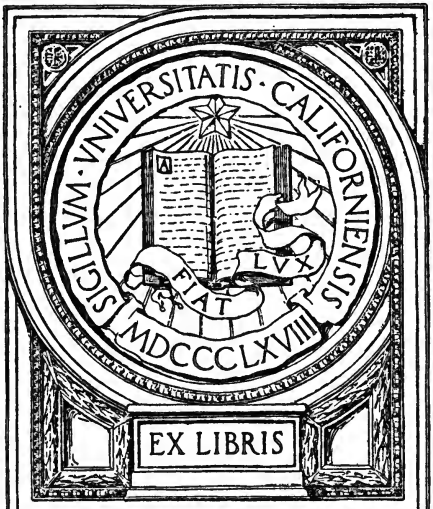

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THE

FLUXIONAL CALCULUS.

AN

ELEMENTARY TREATISE,

IN

TWO VOLUMES,

DESIGNED FOR THE

STUDENTS OF THE UNIVERSITIES,

AND FOR

THOSE WHO DESIRE TO BE ACQUAINTED WITH THE PRINCIPLES OF ANALYSIS.

VOL. II.

CONTAINING THE

CALCULUS OF FLUENTS, FLUXIONAL EQUATIONS, AND THE CALCULUS OF VARIATIONS.

BY THOMAS JEPHSON, B. D.

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ADVERTISEMENT.

This second volume concludes the work which I have undertaken to submit to the publick. I have been compelled to omit several subjects which formed part of my original plan. The discussion of certain curves; a chapter on curve surfaces; the application of the calculus to the finding of the centres of gravity and oscillation, and to some of the branches of natural philosophy; these, together with other subjects which are usually discussed in the elementary treatises which are published in this country, were to have constituted the fourth or miscellaneous part; but I have entirely omitted them, that the work may not swell to an inconvenient size. It is the less necessary at the present day to introduce these subjects into our treatises on pure science, as there have lately appeared publications in every department of the mixed mathematics adapted to the use of students.

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ERRATA.

Page 7, l. 6 from bottom, *for* $\cos.\frac{2(n-1)-\phi}{n}$, *read* $\cos.\frac{2(n-1)\pi-\phi}{n}$.

13, l. 2 from bottom, *for* $\frac{2dx}{n}$, *read* $-\frac{2dx}{n}$.

14, last line, *for* 2, *read* $\frac{2}{n}$.

15, *for* l. 2, *read* $\frac{2}{n}\sin.(n-m)\frac{i\pi}{n}\tan.^{-1}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}$.

35, l. 12, *for* $\sqrt{1-z^2}$, *read* $\sqrt{1-z^{2r}}$.

52, ll. 2 and 3, insert z^{m-2} and z^{m-3} in the denominators.

55, l. 3 from bottom, *for* $\sin.\text{A}$, *read* $\sin.x$, and *for* $\sin.^{2n-2}x$, *read* $\sin.^{n-2}x$.

59, l. 5, *for* $\frac{m+n}{n}$, *read* $\frac{m+n}{2}$.

66, l. 13, *for* A, *read* $m\text{A}$; and l. 15, *for* $\cos.nz$, *read* $\cos.^n z$.

73, l. 8 from bottom, *for* $\frac{m+np}{q}$, *read* $\frac{mq+np}{q}$ and divide the exponents of x in the series by q.

114, l. 23, *for* art. 22, *read* art. 24.

145, l. 5 from bottom, *for* x^4, *read* x.

166, l. 6 from bottom, *for* $\frac{dx}{dy}$, *read* $\frac{da}{dy}$.

179, l. 10, *for* $6x^2p$, *read* $6x^2p^5$.

218, l. 19, *for* a'', *read* a'''.

223, l. 7 from bottom, *for* x, *read* z.

248, l. 6, insert x in the denominator.

281, l. 16 from bottom, *for* equation, *read* equations.

312, l. 6 from bottom, *for* z, *read* z.

315, l. 12, *for* 29, *read* 30.

316, l. 12, *for* v, *read* o.

324, l. 11, *for* 40, *read* 42.

391, l. 3, insert $\therefore$ between x^2 and y.

l. 6, *for* $c'x$, *read* $c'x^5$.

THE

INVERSE CALCULUS.

PART SECOND.

CHAPTER I.

FLUENTS.

1. THE forms of fluents may be divided into three classes, according as the functions which they contain are rational, irrational, or transcendental. It has been shown, Vol. I. Chap. II., that there are two general methods of integrating these, viz.—1st, By the method of continuation; and, 2d, By the method of indeterminate coefficients. By substitution we can frequently reduce the proposed form so that it becomes integrable by one of these methods.

2. *Rational Forms.*

These are to be integrated by the method of indeterminate coefficients. An algebraical process of finding the numerators of the fractions into which the form is to be decomposed has been already given, Chap. II. Art. 60; but by means of the calculus these numerators may be obtained by the following shorter process.

3. *Required to find the partial fractions into which a rational form may be decomposed.*

Let $\frac{P}{Q}$ be the proposed form. The factors of $Q = 0$ are to be found by the theory of equations.

First, suppose all the factors unequal and possible; let one of them be $a + bx$, and let it be required to find the corresponding fraction $\frac{A}{a+bx}$.

Let $\frac{P}{Q} = \frac{A}{a+bx} + \frac{R}{S} = \frac{AS + (a+bx)R}{(a+bx)S}$; whence $R = \frac{P - AS}{a+bx}$. Also, since R is integral, $a+bx$ is a divisor of

$P - AS$; and this obtains whatever value be assigned to x; suppose $a + bx = 0$, or $x = -\frac{a}{b}$; then we shall have

$$P - AS = 0, \text{ or } A = \left(\frac{P}{S}\right) = \left(\frac{P(a+bx)}{Q}\right).$$

Since this value of A is under the form $\frac{0}{0}$, it may be found by differentiating both numerator and denominator, (Vol. I. Ch. V.); and we have

$$A = \left\{\left((a+bx)\frac{dP}{dx} + bP\right)\right\} \div \left(\frac{dQ}{dx}\right) = (bP) \div \left(\frac{dQ}{dx}\right).$$

Similarly, the numerator may be found corresponding to any other denominator.

If *two* factors are equal, S contains $a + bx$; and A, which $= \left(\frac{P}{S}\right)$, $= \infty$; or this process fails; but assuming . . . . $\frac{P}{Q} = \frac{A}{(a+bx)^2} + \frac{A'}{a+bx} + \frac{R}{S}$, we have . . . . . $R = \frac{P - S(A + (a+bx)A')}{(a+bx)^2}$ where s does not contain $a+bx$, and consequently $(a+bx)^2$ is a divisor of $\frac{P}{S} - (A + (a+bx)A')$; wherefore, as before, $\left(\frac{P}{S}\right) = (A + (a+bx)A') = A$; and to find A' we have $\left(\frac{P}{S} - (A + (a+bx)A')\right) = 0$, or . . . . .

$$A' = \left(\frac{\frac{P}{S} - A}{a+bx}\right) = \left(\frac{d \cdot \frac{P}{S}}{bdx}\right).$$

Next, let *three* factors be equal.

Assuming $\frac{P}{Q} = \frac{A}{(a+bx)^3} + \frac{A'}{(a+bx)^2} + \frac{A''}{a+bx} + \frac{R}{S}$, we have $R = \frac{P - S(A + (a+bx)A' + (a+bx)^2A'')}{(a+bx)^3}$, and it may

be shown as before that $A = \left(\frac{P}{S}\right)$, $A' = \frac{1}{b}\left(\frac{d \cdot \frac{P}{S}}{dx}\right)$, ...

$A'' = \frac{1}{1.2b^2}\left(\frac{d^2 \cdot \frac{P}{S}}{dx^2}\right)$.

If there are n factors equal; assuming $\frac{P}{Q} = \frac{A'}{(a+bx)^n} + \frac{A''}{(a+bx)^{n-1}} \cdots + \frac{A^{In}}{a+bx} + \frac{R}{S}$, we shall have . . . . .

$A^{In} = \frac{1}{1.2..(n-1)b^{n-1}} \cdot \left(\frac{d^{n-1} \cdot \frac{P}{S}}{dx^{n-1}}\right)$.

Next, let the fractions contain *one* pair of conjugate roots. These are under the form $(x \mp \alpha)^2 + \beta^2$; or, substituting $\alpha = r \cos. \theta$, $\beta = r \sin. \theta$, $\alpha^2 + \beta^2 = r^2$, they may take the form $x^2 - 2rx \cos. \theta + r^2$.

Assume $\frac{P}{Q} = \frac{Ax + B}{x^2 - 2rx \cos. \theta + r^2} + \frac{R}{S}$; then . . . . . $R = \frac{P - (Ax + B)S}{x^2 - 2rx \cos. \theta + r^2}$. If then we suppose . . . . . $x^2 - 2rx \cos. \theta + r^2 = 0$, or $x = r(\cos. \theta \pm \sqrt{-1} \sin. \theta)$, we shall have as before $\left(\frac{P}{S}\right) = (Ax + B)$. But . . . . . $S = \frac{Q}{x^2 - 2rx \cos. \theta + r^2}$; whence (S), which $= \frac{0}{0}$, ... $= \left(\frac{dQ}{dx}\right) \div (2x - 2r \cos. \theta)$; and, by substitution, $(Ax + B) = (2P(x - r \cos. \theta)) \div \left(\frac{dQ}{dx}\right)$.

Since the possible parts of this equation are equal, and also the impossible parts (Alg. 277), there will arise two equations, by means of which A and B may be determined.

Similarly, if there are $2n$ impossible roots, assuming $\frac{P}{Q} = \frac{A'x + B'}{x^2 - 2rx \cos. \theta + r^2} + \frac{A''x + B''}{(x^2 - 2rx \cos. \theta + r^2)^2} + \cdots$ . . . . .

$+\frac{A'^{n}x + B'^{n}}{(x^2 - 2rx\cos.\theta + r^2)^n} + \frac{R}{S}$, the required coefficients may be determined.

Ex. $\frac{1}{x^3(1+x^2)} = u.$

First, to find the part corresponding to $\frac{1}{x^3}$.

Here $S = 1 + x^2$, $a = 0$, $b = 1$; $\therefore x = 0$, and $P = 1$; $\therefore A = \left(\frac{P}{S}\right) = 1$. Also, $\frac{P}{S} = \frac{1}{1+x^2}$; . . . . . .

$\therefore \frac{d\cdot\frac{P}{S}}{dx} = \frac{-2x}{(1+x^2)^2}$, $\frac{d^2\cdot\frac{P}{S}}{dx^2} = \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}$; . .

$\therefore A' = \left(\frac{d\cdot\frac{P}{S}}{dx}\right) = 0$, and $A'' = \frac{1}{1\cdot 2}\left(\frac{d^2\cdot\frac{P}{S}}{dx^2}\right) = -1$; . . . .

or $u = \frac{1}{x^3} - \frac{1}{x} + \frac{R}{S}$.

Next, to find the part corresponding to $\frac{1}{1+x^2}$.

Here, cos. $\theta = 0$, sin. $\theta = 1$, $r = 1$, $\therefore x = \pm\sqrt{-1}$; $Q = x^3(1+x^2)$, or $\frac{dQ}{dx} = 3x^2(1+x^2) + 2x^4$; $\therefore \left(\frac{dQ}{dx}\right)$. . . $= (2x^4) = 2$; and the equation for determining A and B becomes $\pm\sqrt{-1}\,A + B = \pm\sqrt{-1}$; $\therefore A = 1$, $B = 0$; or the required part is $\frac{x}{1+x^2}$; whence $u = \frac{1}{x^3} - \frac{1}{x} + \frac{x}{1+x^2}$.

4. *Required to integrate* $\frac{x^m dx}{x^n - px^{n-1} + qx^{n-2} - \&c.}$ $= du$; *where* $x^n - px^{n-1} + qx^{n-2} - \&c. = (x-a)(x-b)(x-c)\ldots$ *and* $m < n$.

Assume $\frac{x^m}{x^n - px^{n-1} + qx^{n-2} - \&c.} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$ $+ \ldots \&c.$; and to find the coefficients A, B, C, . . . without the aid of the calculus, we have $x^m = A(x-b)(x-c)\cdots$ $+ B(x-a)(x-c)\cdots + C(x-a)(x-b) + \cdots$

+ &c.; which obtains for all values of x. Let $x = a$; then $a^m = \text{A}(a-b)(a-c)\cdots$, or $\text{A} = \dfrac{a^m}{(a-b)(a-c)\cdots}$. Next, let $x = b$; then $b^m = \text{B}(b-a)(b-c)\cdots$, or $\text{B} = \dfrac{b^m}{(b-a)(b-c)\cdots}$.

Similarly, C, which $= \dfrac{c^m}{(c-a)(c-b)\cdots}$, and all the remaining coefficients may be determined.

Now $du = \dfrac{\text{A}dx}{x-a} + \dfrac{\text{B}dx}{x-b} + \dfrac{\text{C}dx}{x-c} + \cdots$; wherefore

$u = \text{A}l(x-a) + \text{B}l(x-b) + \text{C}l(x-c)\cdots$

It follows, from the original assumption, that m must be less than n; for the resulting value of x^m, viz.— $\text{A}(x-b)(x-c)\cdots + \text{B}(x-a)(x-c)\cdots + \text{C}(x-a)(x-b)\cdots +$ &c. cannot rise to more than $n-1$ dimensions, be the coefficients A, B, C $\cdots$ what they may.

The cases of equal and of impossible roots may be provided for as in Vol. I. Ch. II. Arts. 60 and 61.

5. *Required to integrate* $\dfrac{x^m dx}{x^n - px^{n-1} + qx^{n-2} - \text{&c.}} = du$, *when* m *is not less than* n.

Here assume $\dfrac{1}{x^n - px^{n-1} + qx^{n-2} - \text{&c.}} = \dfrac{\text{A}}{x-a} + \dfrac{\text{B}}{x-b} + \dfrac{\text{C}}{x-c} + \text{&c.}$; then $\text{A} = \dfrac{1}{(a-b)(a-c)\cdots}$, $\cdots\cdots$ $\text{B} = \dfrac{1}{(b-a)(b-c)\cdots}$, $\text{C} = \dfrac{1}{(c-a)(c-b)\cdots}$, &c. and we have

$$du = \frac{\text{A}x^m dx}{x-a} + \frac{\text{B}x^m dx}{x-b} + \frac{\text{C}x^m dx}{x-c} + \text{&c.}$$

$$= \text{A}dx\left\{x^{m-1} + ax^{m-2} + a^2x^{m-3}\cdots + \frac{a^m}{x-a}\right\}$$

$$+ \text{B}dx\left\{x^{m-1} + bx^{m-2} + b^2x^{m-3}\cdots + \frac{b^m}{x-b}\right\}$$

$$+ \text{&c.}$$

$$\text{Whence } u = \text{A}\left\{\frac{x^m}{m} + \frac{ax^{m-1}}{m-1} + \frac{a^2x^{m-2}}{m-2} \cdots + a^m l(x-a)\right\}$$
$$+ \text{B}\left\{\frac{x^m}{m} + \frac{bx^{m-1}}{m-1} + \frac{b^2x^{m-2}}{m-2} \cdots + b^m l(x-b)\right\}$$
$$+ \&\text{c.}$$
$$= \frac{x^m}{m}(\text{A} + \text{B} + \cdots) + \frac{x^{m-1}}{m-1}(a\text{A} + b\text{B} \cdots)$$
$$+ \frac{x^{m-2}}{m-2}(a^2\text{A} + b^2\text{B} \cdots) + \&\text{c.}$$
$$+ a^m\text{A}l(x-a) + b^m\text{B}l(x-b) + \&\text{c.}$$
$$\text{by substitution} = \text{A}'x^m + \text{B}'x^{m-1} + \text{C}'x^{m-2} + \cdots + \text{E}'x^{m-n+1} + \cdots$$
$$+ a^m\text{A}l(x-a) + b^m\text{B}l(x-b) + \&\text{c.}$$

Differentiate this value of u, and equate it with the proposed form; and there results

$$\frac{x^m}{x^n - px^{n-1} + qx^{n-2} - \&\text{c.}} = m\text{A}'x^{m-1} + (m-1)\text{B}'x^{m-2} \cdots$$
$$+ (m-n+1)\text{E}'x^{m-n} + \cdots$$
$$+ \frac{a^m\text{A}}{x-a} + \frac{b^m\text{B}}{x-b} + \&\text{c.}$$

Clearing this of fractions, we have an equation of the form

$$x^m = a'\text{A}'x^{m+n-1} + (b'\text{A}' + c'\text{B}')x^{m+n-2}$$
$$+ (e'\text{A}' + f'\text{B}' + g'\text{C}')x^{m+n-3} + \cdots$$
$$+ (k'\text{A}' + l'\text{B}' \cdots + m'\text{E}')x^m + \cdots$$
$$+ a^m\text{A}(x-b)(x-c)\cdots + b^m\text{B}(x-a)(x-c)\cdots$$
$$+ c^m\text{C}(x-a)(x-b)\cdots + \&\text{c.}$$

This equation is to obtain whatever be the value of x; and consequently (Alg. 347) all the coefficients before that of x^m are to $= 0$; or the first $n-1$ terms A′, B′, C′, &c. each $= 0$; and we have

$$u = \text{E}'x^{m-n+1} + \cdots$$
$$+ a^m\text{A}l(x-a) + b^m\text{B}l(x-b) + c^m\text{C}l(x-c) + \cdots$$

Cor. When m is less than n, all the coefficients A′, B′, C′ $\cdots = 0$; and we have $u = a^m\text{A}l(x-a) + b^m\text{B}l(x-b) + c^m\text{C}l(x-c) + \cdots$, which agrees with the preceding article.

If the theory of algebraical equations were perfect, it is obvious that all rational forms would be integrable by the above method; since whatever be the numerator, the pro-

posed form may be separated into parts which shall coincide with either this or the preceding article. But there is no known formula by which an equation of n dimensions may be resolved into its factors; and, in consequence, particular methods have been invented by Cotes, De Moivre, and Waring, for the purpose of exhibiting the fluents of certain rational forms; some of these we shall now proceed to give.

6. *Required to resolve the trinomial* $x^{2n} - 2kx^n + 1 = 0$ *into its quadratick divisors, where* k *is not greater than unity.*

Assume $k = \cos.\theta$; (then vol. 1, ch. 2, 37, cor. 1.) if $2\cos.\frac{\theta}{n} = x + \frac{1}{x}$, we have $2\cos.\theta = x^n + \frac{1}{x^n}$; i. e. if . . . . $x^2 - 2x\cos.\frac{\theta}{n} + 1 = 0$, then shall $x^{2n} - 2x^n\cos.\theta + 1 = 0$; or the roots of the former equation are two of the roots of the latter.

Now, $\cos.\theta = \cos.(2\pi + \theta) = \cos.(4\pi + \theta)$ . . . . . $= \cos.(2(n-1)\pi + \theta)$ (1); wherefore $\cos.\frac{\theta}{n}$ has n *different* values corresponding to the same value of θ. It cannot have more; for taking the next value of $\cos.\theta$, viz. $\cos.(2n\pi + \theta)$, there results $\cos.\left(2\pi + \frac{\theta}{n}\right)$, which $= \cos.\frac{\theta}{n}$; or the same values recur after the nth.

Since $\cos.\theta$ also $= \cos.(2\pi - \theta) = \cos.(4\pi - \theta)$ . . . . $= \cos.(2(n-1)\pi - \theta)$ (2); from these there will likewise result n different values of $\cos.\frac{\theta}{n}$; but they are the same as the former taken in a reverse order; for $\cos.\frac{2(n-1)\pi - \theta}{n}$ . . . $= \cos.\left(2\pi - \frac{2\pi + \theta}{n}\right) = \cos.\frac{2\pi + \theta}{n}$.

We obtain then n different quadraticks which are the divisors of the proposed trinomial; and substituting successively the terms of series (2) for those of series (1) taken in a reverse order, we have $x^{2n} - 2x^n\cos.\theta + 1$ . . . .

$$= \left(x^2 - 2x\cos.\frac{\theta}{n} + 1\right) \times \ldots$$

$$\left(x^2 - 2x\cos.\frac{2\pi+\theta}{n} + 1\right)\left(x^2 - 2x\cos.\frac{2\pi-\theta}{n} + 1\right) \times \ldots$$

$$\left(x^2 - 2x\cos.\frac{4\pi+\theta}{n} + 1\right)\left(x^2 - 2x\cos.\frac{4\pi-\theta}{n} + 1\right) \times \cdots$$

to n terms.

Cor. 1. $x^{2n} - 2r^n x^n \cos.\theta + r^{2n} = \left(x^2 - 2rx\cos.\frac{\theta}{n} + r^2\right)$

$$\times\left(x^2 - 2rx\cos.\frac{2\pi+\theta}{n} + r^2\right)\left(x^2 - 2rx\cos.\frac{2\pi-\theta}{n} + r^2\right) \times \ldots$$

Here the roots of the proposed trinomial are multiplied by r (Alg. 286); or the radius is transformed from rad. $= 1$ to rad. $= r$ (Trig. p. 19).

Cor. 2. The form of the continued product will be different according as n is even or odd.

When n is even, the last factor, which is the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term of the series whose first term is $x^2 - 2x\cos.\frac{\theta}{n} + 1$, is $x^2 - 2x\cos.\frac{n\pi+\theta}{n} + 1$, or $x^2 + 2x\cos.\frac{\theta}{n} + 1$; and consequently we have

$x^{2n} - 2x^n\cos.\theta + 1 =$

$$\left(x^2 - 2x\cos.\frac{\theta}{n} + 1\right)\left(x^2 + 2x\cos.\frac{\theta}{n} + 1\right) \times \ldots\ldots$$

$$\left(x^2 - 2x\cos.\frac{2\pi+\theta}{n} + 1\right)\left(x^2 - 2x\cos.\frac{2\pi-\theta}{n} + 1\right) \times \cdots$$

$$\left(x^2 - 2x\cos.\frac{4\pi+\theta}{n} + 1\right)\left(x^2 - 2x\cos.\frac{4\pi-\theta}{n} + 1\right) \times \cdots$$

When n is odd and $= 2m + 1$, there is one divisor $x^2 - 2x\cos.\frac{\theta}{n} + 1$, and m double divisors such as

$$\left(x^2 - 2x\cos.\frac{2\pi+\theta}{n} + 1\right)\left(x^2 - 2x\cos.\frac{2\pi-\theta}{n} + 1\right).$$

Cor. 3. If θ be supposed $= 0$, $\cos.\theta = \cos.\frac{\theta}{n} = 1$; and

we have, if n is even, $x^{2n} - 2r^n x^n + r^{2n} =$

$$(x^2 - 2rx + r^2)(x^2 + 2rx + r^2)\left(x^2 - 2rx\cos.\frac{2\pi}{n} + r^2\right)^2 \cdots$$

$$\left(x^2 - 2rx\cos.\frac{4\pi}{n} + r^2\right)^2\left(x^2 - 2rx\cos.\frac{6\pi}{n} + r^2\right)^2 \ldots \text{ or,}$$

$x^n - r^n =$

$$(x^2 - r^2)\left(x^2 - 2rx\cos.\frac{2\pi}{n} + r^2\right)\left(x^2 - 2rx\cos.\frac{4\pi}{n} + r^2\right)\cdots$$

If n is odd, $x^n - r^n =$

$$(x - r)\left(x^2 - 2rx\cos.\frac{2\pi}{n} + r^2\right)\left(x^2 - 2rx\cos.\frac{4\pi}{n} + r^2\right)\cdot \cdots$$

Cor. 4. Suppose $\theta = \pi$; then k or $\cos.\theta = -1$, and we have from the article, if n is even,

$$x^{2n} + 2r^n x^n + r^{2n} = \left(x^2 - 2rx\cos.\frac{\pi}{n} + r^2\right)^2 \cdot \cdots$$

$$\left(x^2 - 2rx\cos.\frac{3\pi}{n} + r^2\right)^2\left(x^2 - 2rx\cos.\frac{5\pi}{n} + r^2\right)^2, \ldots \text{ or}$$

$$x^n + r^n = \left(x^2 - 2rx\cos.\frac{\pi}{n} + r^2\right)\left(x^2 - 2rx\cos.\frac{3\pi}{n} + r^2\right)$$

$$\left(x^2 - 2rx\cos.\frac{5\pi}{n} + r^2\right)\cdots$$

If n is odd, $x^n + r^n =$

$$(x + r)\left(x^2 - 2rx\cos.\frac{\pi}{n} + r^2\right)\left(x^2 - 2rx\cos.\frac{3\pi}{n} + r^2\right)\cdots$$

Cor. 5. Similarly, by supposing $\theta = \frac{\pi}{2}$, the quadratick divisors of $x^{2n} + r^{2n} = 0$ may be found.

Cor. 6. If k be *greater* than unity, the trinomial by the solution of a quadratick is resolved into $(x^n - k + \sqrt{k^2 - 1})$ $(x^n - k - \sqrt{k^2 - 1}) = 0$, or $(x^n - r^n)\,(x^n - r'^n) = 0$, when r and r' are both possible; and the fraction $\frac{1}{x^{2n} - 2kx^n + 1}$ is resolved into $\frac{1}{r^n - r'^n}\left\{\frac{1}{x^n - r^n} - \frac{1}{x^n - r'^n}\right\}$.

Cor. 7. The double divisors, reckoning from the end, are the same as those from the beginning, the sign of the middle term being changed from $-$ to $+$.

For the $\left(\frac{n}{2}-1\right)^{\text{th}}$ contains $\frac{\left(\frac{n}{2}-1\right)2\pi \pm \theta}{n}$ which $= \cos.\left(\pi - \frac{2\pi \mp \theta}{n}\right) = -\cos.\frac{2\pi \mp \theta}{n}$, which are the same as those in the first double divisor with their sign changed; hence, if n be even, we have the form

$$x^{2n} - 2r^n x^n \cos.\theta + r^{2n} =$$

$$\left(x^2 - 2rx\cos.\frac{\theta}{n} + r^2\right)\left(x^2 + 2rx\cos.\frac{\theta}{n} + r^2\right) \times \ \ldots$$

$$\left(x^2 - 2rx\cos.\frac{2\pi+\theta}{n} + r^2\right)\left(x^2 + 2rx\cos.\frac{2\pi+\theta}{n} + r^2\right) \times \ldots$$

$$\left(x^2 - 2rx\cos.\frac{4\pi+\theta}{n} + r^2\right)\left(x^2 + 2rx\cos.\frac{4\pi+\theta}{n} + r^2\right) \times \ldots$$

7. *Required to integrate* $\frac{dx}{1-x^n} = du$ *when* n *is even.*

Here $1 - x^n = (1-x^2)(1 - 2ax + x^2)(1 - 2bx + x^2) \ldots$ where $a = \cos.\frac{2\pi}{n}$, $b = \cos.\frac{4\pi}{n}$ . . . (Art. 6, Cor. 3.)

Whence $\frac{nx^{n-1}}{1-x^n} = \frac{2x}{1-x^2} + \frac{2a-2x}{1-2ax+x^2} + \frac{2b-2x}{1-2bx+x^2} + \ldots$ or multiplying by x,

$$\frac{nx^n}{1-x^n} = \frac{2x^2}{1-x^2} + \frac{2ax-2x^2}{1-2ax+x^2} + \frac{2bx-2x^2}{1-2bx+x^2}.$$

Add n to the left side of this equation, and 2 to each term on the right, which will not destroy the equation, and there results

$$\frac{n}{1-x^n} = \frac{2}{1-x^2} + \frac{2-2ax}{1-2ax+x^2} + \frac{2-2bx}{1-2bx+x^2} + \ldots \text{ whence}$$

$$du = \frac{2dx}{n}\left\{\frac{1}{1-x^2} + \frac{1-ax}{1-2ax+x^2} + \frac{1-bx}{1-2bx+x^2} + \ldots\right\}$$

and integrating according to the form, vol. i. c. ii. 61, we have

$$u=\frac{1}{n}l\frac{1+x}{1-x}-\frac{a}{n}l(1-2ax+x^2)+\frac{2a'}{n}\text{tan.}^{-1}\frac{x-a}{a'}\Bigg\}\begin{matrix}\text{where}\\ a'=\text{sin.}\frac{2\pi}{n},\end{matrix}$$
$$-\frac{b}{n}l(1-2bx+x^2)+\frac{2b'}{n}\text{tan.}^{-1}\frac{x-b}{b'}\Bigg\}\ b'=\text{sin.}\frac{4\pi}{n}.$$
. . . . . .

8. *Required to integrate* $\frac{dx}{1+x^n}=du$, *when n is odd.*

Here $1+x^n=(1+x)(1-2ax+x^2)(1-2bx+x^2)\ \cdot\ \cdot\ \cdot$

where $a=\text{cos.}\frac{\pi}{n}$, $b=\text{cos.}\frac{3\pi}{n}$, $\cdot\ \cdot\ \cdot$ (Art. 6, Cor. 4.)

Whence $\frac{nx^{n-1}}{1+x^n}=\frac{1}{1+x}+\frac{2x-2a}{1-2ax+x^2}+\frac{2x-2b}{1-2bx+x^2},\cdots$

or $\frac{nx^n}{1+x^n}=\frac{x}{1+x}+\frac{2x^2-2ax}{1-2ax+x^2}+\frac{2x^2-2bx}{1-2bx+x^2}\cdots$

Let $n=2m+1$; then since there are m terms with quadratick divisors (Art. 6, Cor. 2.), if we subtract the left side of the equation from n, the first term on the right from 1, and each of the remaining terms from 2, the equation will not be destroyed; and there results

$$\frac{n}{1+x^n}=\frac{1}{1+x}-\frac{2ax-2}{1-2ax+x^2}-\frac{2bx-2}{1-2bx+x^2}\ \cdot\ \cdot\ \cdot\ \text{or}$$

$$du=\frac{dx}{n(1+x)}-\frac{2dx}{n}\left\{\frac{ax-1}{1-2ax+x^2}+\frac{bx-1}{1-2bx+x^2}\ \cdot\ \cdot\ \cdot\right\}$$

and integrating,

$$u=l.(1+x)^{\frac{1}{n}}-\frac{a}{n}l(1-2ax+x^2)+\frac{2a'}{n}\text{tan.}^{-1}\frac{x-a}{a'}\Bigg\}\begin{matrix}\text{where}\\ a'=\text{sin.}\frac{\pi}{n},\end{matrix}$$
$$-\frac{b}{n}l(1-2bx+x^2)+\frac{2b'}{n}\text{tan.}^{-1}\frac{x-b}{b'}\Bigg\}\ b'=\text{sin.}\frac{3\pi}{n}.$$
. . . . . .

In order to correct these fluents from $x=0$, add to $\text{tan.}^{-1}\frac{x-a}{a'}$ the constant quantity $\text{tan.}^{-1}\frac{a}{a'}$; then their sum

$$=\text{tan.}^{-1}\frac{\frac{x}{a'}}{1-\frac{ax-a^2}{a'^2}}=\text{tan.}^{-1}\frac{a'x}{1-ax},$$ which begins from

$x=0$; and similarly $\text{tan.}^{-1}\frac{x-b}{b'}$, when corrected . . . .

$= \tan.^{-1}\frac{b'x}{1-bx}$; which must therefore be substituted in the expression for u.

Similarly, we may integrate and exhibit $\frac{dx}{1-x^n}$ when n is odd, and $\frac{dx}{1+x^n}$ when n is even.

9. *Required to integrate* $\frac{x^m dx}{1+x^n} = du$, *when* $m < n$.

First, let n be *even*; then

$1 + x^n = (1 - 2ax + x^2)\,(1 - 2bx + x^2)\,(1 - 2cx + x^2) \cdot \cdot \cdot$

where $a = \cos.\frac{\pi}{n}$, $b = \cos.\frac{3\pi}{n}$, $c = \cos.\frac{5\pi}{n} \cdot \cdot \cdot$

Whence $\frac{nx^{n-1}}{1+x^n} = \frac{2x-2a}{1-2ax+x^2} + \frac{2x-2b}{1-2bx+x^2} + \cdot \cdot \cdot$

To find the first term of u; we have the *coefficient* of the first term of $du = \frac{x^m}{nx^{n-1}}(2x - 2a)$; which $\cdot \cdot \cdot$

$= \frac{1}{nx^{n-m-1}}\left\{x - \frac{1}{x}\right\} \cdot \cdot \cdot$

$= \frac{x}{n}\left\{x^{n-m-1} + \frac{1}{x^{n-m-1}}\right\} \cdot \cdot \cdot$

$-\frac{1}{n}\left\{x^{n-m} + \frac{1}{x^{n-m}}\right\} = -\frac{2}{n}(hx - g)$, where

$h = -\cos.(n - m - 1)\frac{\pi}{n} = \cos.(m + 1)\frac{\pi}{n}$, and

$g = -\cos.(n - m)\frac{\pi}{n} = \cos.m\frac{\pi}{n}$; or the first term of du

$= -\frac{2dx}{n}\,\frac{hx-g}{1-2ax+x^2}$; whose fluent

$= -\frac{2}{n}\left\{h.l\sqrt{1-2ax+x^2} - \frac{g-ah}{a'}\tan.^{-1}\frac{x-a}{a'}\right\}$ if $a' = \sin.\frac{\pi}{n}$

$= -\frac{h}{n}l(1-2ax+x^2) + \frac{2h'}{n}\tan.^{-1}\frac{x-a}{a'}$, for $g = ah + a'h'$

$= -\frac{h}{n}l(1-2ax+x^2) + \frac{2h'}{n}\tan.^{-1}\frac{a'x}{1-ax}$, when corrected from $x = 0$.

Similarly, the second term of $du = -\frac{2dx}{n}\,\frac{hx-g}{1-2bx+x^2}$

where $h = \cos.(m+1)\frac{3\pi}{n}$ and $g = \cos.m\frac{3\pi}{n}$, which may be integrated as before, and thus the whole fluent may be exhibited.

Next let n be odd.

Here $1+x^n=(1+x)(1-2ax+x^2)(1-2bx+x^2)\cdots$; and consequently all the terms are the same as before, except the first.

The coefficient of the first term of $du=\frac{x^m}{nx^{n-1}}$; but here the factor $1+x=0$ gives $x=-1$, and consequently the coefficient $=\frac{(-1)^m}{n}$; and integrating, the first term $\cdots$ $=\frac{(-1)^m}{n}l(1+x)$.

10. *Required to integrate* $\frac{x^m dx}{1-x^n}=du$, *when* $m<n$.

First, let n be *even*, then

$1-x^n=(1-x^2)(1-2ax+x^2)(1-2bx+x^2)\ldots$

where $a=\cos.\frac{2\pi}{n}$, $b=\cos.\frac{4\pi}{n}$, $\ldots$

Whence $\frac{nx^{n-1}}{1-x^n}=\frac{2x}{1-x^2}+\frac{2a-2x}{1-2ax+x^2}+\frac{2b-2x}{1-2bx+x^2}+\ldots$

Let the *second* term of u be required; then the coefficient of the second term of $du=\frac{x^m}{nx^{n-1}}(2a-2x)$; which $\ldots$.

$$=\frac{1}{nx^{n-m-1}}\left\{\frac{1}{x}-x\right\}\ \ldots$$

$$=\frac{1}{n}\left\{x^{n-m}+\frac{1}{x^{n-m}}\right\}-\frac{x}{n}\left\{x^{n-m-1}+\frac{1}{x^{n-m-1}}\right\}\ \ldots$$

$$=\frac{2}{n}(g-hx)\text{ if }g=\cos.(n-m)\frac{2\pi}{n}=\cos.m\frac{2\pi}{n},\ \ldots$$

and $h=\cos.(n-m-1)\frac{2\pi}{n}=\cos.(m+1)\frac{2\pi}{n}$; $\ldots$

or the second term of $du=\frac{2dx}{n}\ \frac{hx-g}{1-2ax+x^2}$; whose

fluent $=-\frac{2}{n}\left\{h.l\sqrt{1-2ax+x^2}-\frac{g-ah}{a'}\tan.^{-1}\frac{x-a}{a'}\right\}$

$= -\frac{h}{n} l(1-2ax+x^2) + \frac{2h'}{n} \tan.^{-1} \frac{x-a}{a'}$, for $g = ah + a'h'$

$= -\frac{h}{n} l(1-2ax+x^2) + \frac{2h'}{n} \tan.^{-1} \frac{a'x}{1-ax}$, when corrected from $x = 0$.

Similarly, the third term of $du = -\frac{2dx}{n} \cdot \frac{hx-g}{1-2bx+x^2}$, where $g = \cos.m\frac{4\pi}{n}$ and $h = \cos.(m+1)\frac{4\pi}{n}$, which may be integrated as before; and so on.

To integrate the first term, we have its coefficient $= \frac{x^m}{nx^{n-1}} 2x$; but $1 - x^2 = (1 - x)(1 + x) = 0$; which gives $x = +1$ and $x = -1$; wherefore, when decomposed, it is $\frac{dx}{n}\left\{\frac{(+1)^m}{1-x} + \frac{(-1)^m}{1+x}\right\}$; the fluent of which is the first term of u; which therefore $= \frac{1}{n} l \frac{1+x}{1-x}$ or $-\frac{1}{n} l(1-x^2)$, according as m is even or odd.

Next let n be *odd*; then

$1 - x^n = (1 - x)(1 - 2ax + x^2)(1 - 2bx + x^2) \ldots$ and consequently all the terms are the same as before, except the first; and its coefficient is $\frac{x^m}{nx^{n-1}}$; but $1 - x = 0$ gives $x = 1$; wherefore the first term $= \int \frac{dx}{n} \cdot \frac{1}{1-x} \ldots$

$= -\frac{1}{n} l(1 - x)$.

11. $du = \frac{x^{m-1} + x^{n-m-1}}{1+x^n} dx$.

It appears from Art. 9, that any term of u, when n is even, may be represented by

$$-\frac{1}{n}\cos.m\frac{i\pi}{n} l\left(1 - 2x\cos.\frac{i\pi}{n} + x^2\right) + 2\sin.m\frac{i\pi}{n}\tan.^{-1}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}$$

$$\left.\begin{aligned} &-\frac{1}{n}\cos.(n-m)\frac{i\pi}{n}l\left(1-2x\cos.\frac{i\pi}{n}+x^2\right)\\ &+2\sin.(n-m)\frac{i\pi}{n}\tan.^{-1}\frac{x\sin.i\pi}{1-x\cos.\frac{i\pi}{n}}\end{aligned}\right\}$$

where i is any odd number not greater than n.

But $-\frac{1}{n}\cos.(n-m)\frac{i\pi}{n} = -\frac{1}{n}\cos.(i\pi - m\frac{i\pi}{n})\ldots$

$=\frac{1}{n}\cos.m\frac{i\pi}{n}$; or the logarithmick parts destroy each other, and the term may be represented by the circular function

$$\frac{4}{n}\sin.m\frac{i\pi}{n}\tan.^{-1}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}.$$

If n be odd, then the whole of the first term

$=\frac{(-1)^{m-1}}{n}l(1+x)+\frac{(-1)^{n-m-1}}{n}l(1+x)$ which $=0$; and consequently in either case u may be exhibited as a circular function free from logarithms, by substituting in the above expression the odd numbers 1, 3, 5, . . . for i.

Thus, (1.) $\int\frac{x^{m-1}+x^{n-m-1}}{1+x^n}dx$

$$=\frac{4}{n}\sin.m\frac{\pi}{n}\tan.^{-1}\frac{x\sin.\frac{\pi}{n}}{1-x\cos.\frac{\pi}{n}}$$

$$+\frac{4}{n}\sin.m\frac{3\pi}{n}\tan.^{-1}\frac{x\sin.\frac{3\pi}{n}}{1-x\cos.\frac{3\pi}{n}}$$

. . . . . .

$$+\frac{4}{n}\sin.m\frac{i\pi}{n}\tan.^{-1}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}$$

Similarly, the following forms may be demonstrated.

$$(2.)\int \frac{x^{m-1}-x^{n-m-1}}{1+x^n}\,dx$$

$$=-\frac{2}{n}\cos.m\frac{\pi}{n}\,l(1-2x\cos.\frac{\pi}{n}+x^2)$$
$$-\frac{2}{n}\cos.m\frac{3\pi}{n}\,l(1-2x\cos.\frac{3\pi}{n}+x^2)$$
$$\cdots\cdots\cdots\cdots$$
$$-\frac{2}{n}\cos.m\frac{i\pi}{n}\,l(1-2x\cos.\frac{i\pi}{n}+x^2)$$

where i is the greatest *odd* integer not greater than n.

When n is odd, $i=n$; and the last term becomes $-\frac{2}{n}\cos.m\pi l.(1+x)^2=\mp\frac{4}{n}l(1+x)$ according as m is even or odd.

$$(3.)\int \frac{x^{m-1}-x^{n-m-1}}{1-x^n}\,dx$$

$$=\frac{4}{n}\sin.m\frac{2\pi}{n}\tan.^{-1}\frac{x\sin.\frac{2\pi}{n}}{1-x\cos.\frac{2\pi}{n}}$$
$$+\frac{4}{n}\sin.m\frac{4\pi}{n}\tan.^{-1}\frac{x\sin.\frac{4\pi}{n}}{1-x\cos.\frac{4\pi}{n}}$$
$$\cdots\cdots\cdots\cdots$$
$$+\frac{4}{n}\sin.m\frac{i\pi}{n}\tan.^{-1}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}$$

where i is the greatest *even* integer not greater than n.

$$(4.)\int \frac{x^{m-1}+x^{n-m-1}}{1-x^n}\,dx$$

$$=-\frac{2}{n}\,l(1-x)$$
$$-\frac{4}{n}\cos.m\frac{2\pi}{n}\,l(1-2x\cos.\frac{2\pi}{n}+x^2)$$

$$-\frac{4}{n}\cos.m\frac{4\pi}{n}l(1-2x\cos.\frac{4\pi}{n}+x^2)$$

. . . . . .

$$-\frac{4}{n}\cos.m\frac{i\pi}{n}l(1-2x\cos.\frac{i\pi}{n}+x^2).$$

12. EXAMPLES.

1. $\int\frac{dx}{1+x}=l(1+x).$

2. $\int\frac{dx}{1+x^3}=\frac{1}{3}l(1+x)-\frac{1}{3}\cos.\frac{\pi}{3}l(1-2x\cos.\frac{\pi}{3}+x^2)$

$$+\frac{2}{3}\sin.\frac{\pi}{3}\tan.^{-1}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$$

$$=\frac{1}{3}l.\frac{1+x}{\sqrt{1-x+x^2}}+\frac{1}{\sqrt{3}}\tan.^{-1}\frac{x\sqrt{3}}{2-x}.$$

3. $\int\frac{dx}{1+x^4}=-\frac{1}{4}\cos.\frac{\pi}{4}l(1-2x\cos.\frac{\pi}{4}+x^2)$

$$+\frac{1}{2}\sin.\frac{\pi}{4}\tan.^{-1}\frac{x\sin.\frac{\pi}{4}}{1-x\cos.\frac{\pi}{4}}$$

$$-\frac{1}{4}\cos.\frac{3\pi}{4}l(1-2x\cos.\frac{3\pi}{4}+x^2)$$

$$+\frac{1}{2}\sin.\frac{3\pi}{4}\tan.^{-1}\frac{x\sin.\frac{3\pi}{4}}{1-x\cos.\frac{3\pi}{4}}$$

$$=\frac{1}{4\sqrt{2}}l\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}+\frac{1}{2\sqrt{2}}\tan.^{-1}\frac{x\sqrt{2}}{1-x^2}.$$

4. $\int\frac{dx}{(1+x)^5}=\frac{1}{5}l(1+x)-\frac{1}{5}\cos.\frac{\pi}{5}l(1-2x\cos.\frac{\pi}{5}+x^2)$

$$+\frac{2}{5}\sin.\frac{\pi}{5}\tan.^{-1}\frac{x\sin.\frac{\pi}{5}}{1-x\cos.\frac{\pi}{5}}$$

$$-\frac{1}{5}\cos.\frac{3\pi}{5}l(1-2x\cos.\frac{3\pi}{5}+x^2)$$

$$+\tfrac{2}{5}\sin.\frac{3\pi}{5}\tan.^{-1}\frac{x\sin.\dfrac{3\pi}{5}}{1-x\cos.\dfrac{3\pi}{5}}.$$

5. $\int\frac{dx}{1+x^6}=\frac{1}{4\sqrt{3}}\,l\,\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}+\frac{1}{6}\tan.^{-1}\frac{3x(1-x^2)}{1-4x^2+x^4}.$

6. $\int\frac{dx}{1-x^3}=-\frac{1}{3}l(1-x)-\frac{1}{3}\cos.\frac{2\pi}{3}\,l(1-2x\cos.\frac{2\pi}{3}+x^2)$

$$+\tfrac{2}{3}\sin.\frac{2\pi}{3}\tan.^{-1}\frac{x\sin.\dfrac{2\pi}{3}}{1-x\cos.\dfrac{2\pi}{3}}.$$

7. $\int\frac{dx}{1-x^4}=\frac{1}{4}l\frac{1+x}{1-x}-\frac{1}{4}\cos.\frac{\pi}{2}\,l(1-2x\cos.\frac{\pi}{2}+x^2)$

$$+\tfrac{1}{2}\sin.\frac{\pi}{2}\tan.^{-1}\frac{x\sin.\dfrac{\pi}{2}}{1-x\cos.\dfrac{\pi}{2}}$$

$$=\tfrac{1}{4}l\frac{1+x}{1-x}+\tfrac{1}{2}\tan.^{-1}x.$$

8. $\int\frac{dx}{1-x^6}=\frac{1}{6}l\frac{1+x}{1-x}-\frac{1}{6}\cos.\frac{\pi}{3}\,l(1-2x\cos.\frac{\pi}{3}+x^2)$

$$+\tfrac{1}{3}\sin.\frac{\pi}{3}\tan.^{-1}\frac{x\sin.\dfrac{\pi}{3}}{1-x\cos.\dfrac{\pi}{3}}$$

$$-\tfrac{1}{6}\cos.\frac{2\pi}{3}\,l(1-2x\cos.\frac{2\pi}{3}+x^2)$$

$$+\tfrac{1}{3}\sin.\frac{2\pi}{3}\tan.^{-1}\frac{x\sin.\dfrac{2\pi}{3}}{1-x\cos.\dfrac{2\pi}{3}}$$

$$=\tfrac{1}{6}l\frac{1+x}{1-x}+\tfrac{1}{12}l\frac{1+x+x^2}{1-x+x^2}+\frac{1}{2\sqrt{3}}\tan.^{-1}\frac{x\sqrt{3}}{1-x^2}.$$

9. $\int\frac{x^3dx}{1-x^4}=-\frac{1}{4}l(1-x^2)-\frac{1}{4}l(1+x^2)=-\frac{1}{4}l(1-x^4).$

$$10.\ \int\frac{x^3dx}{1-x^6}=-\tfrac{1}{6}l(1-x^2)+\tfrac{1}{6}\cos.\frac{\pi}{3}l(1-2x\cos.\frac{\pi}{3}+x^2)$$

$$-\tfrac{1}{3}\sin.\frac{\pi}{3}\tan.^{-1}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$$

$$+\tfrac{1}{6}\cos.\frac{\pi}{3}l(1+2x\cos.\frac{\pi}{3}+x^2)$$

$$+\tfrac{1}{3}\sin.\frac{\pi}{3}\tan.^{-1}\frac{x\sin.\frac{\pi}{3}}{1+x\cos.\frac{\pi}{3}}$$

$$=-\tfrac{1}{6}l(1-x^2)+\tfrac{1}{12}l(1+x^2+x^4)-\frac{1}{2\sqrt{3}}\tan.^{-1}\frac{x^2\sqrt{3}}{2+x^2}$$

$$11.\ \int\frac{1+x^2}{1+x^4}dx=\sin.\frac{\pi}{4}\tan.^{-1}\frac{x\sin.\frac{\pi}{4}}{1-x\cos.\frac{\pi}{4}}$$

$$+\sin.\frac{3\pi}{4}\tan.^{-1}\frac{x\sin.\frac{3\pi}{4}}{1-x\cos.\frac{3\pi}{4}}$$

$$=\frac{1}{\sqrt{2}}\tan.^{-1}\frac{x\sqrt{2}}{1-x^2}.$$

$$12.\ \int\frac{1+x^4}{1+x^6}dx=\tfrac{1}{3}\tan.^{-1}\frac{x}{1-x^2}+\tfrac{2}{3}\tan.^{-1}x$$

$$=\tfrac{1}{3}\tan.^{-1}\frac{x}{1-x^2}+\tfrac{1}{3}\tan.^{-1}\frac{2x}{1-x^2}$$

$$=\tfrac{1}{3}\tan.^{-1}\frac{3x(1-x^2)}{1-4x^2+x^4}.$$

13. *Required to resolve* $\frac{1}{1-2kx^n+x^{2n}}$ *into fractions whose denominators shall be the quadratick divisors of the trinomial.*

Assume as before $k=\cos.\theta$; and let $a, b, c, \ldots$ $=\cos.\frac{\theta}{n}, \cos.\frac{2\pi+\theta}{n}, \cos.\frac{4\pi+\theta}{n}, \ldots$ then $1-2kx^n+x^{2n}$ $=(1-2ax+x^2)(1-2bx+x^2)(1-2cx+x^2)\ldots$

and consequently $\frac{2nkx^{n-1}-2nx^{2n-1}}{1-2kx^n+x^{2n}} = \ldots\ldots$

$\frac{2a-2x}{1-2ax+x^2} + \frac{2b-2x}{1-2bx+x^2} + \ldots\ldots\ldots$ or

$\frac{2nkx^n-2nx^{2n}}{1-2kx^n+x^{2n}} = \frac{2ax-2x^2}{1-2ax+x^2} + \frac{2bx-2x^2}{1-2bx+x^2} + \ldots$ to n terms.

Add n to the left side of this equation, and 1 to each term on the right; and there results

$\frac{n(1-x^{2n})}{1-2kx^n+x^{2n}} = \frac{1-x^2}{1-2ax+x^2} + \frac{1-x^2}{1-2bx+x^2} + \ldots\ldots$ or

$\frac{1}{1-2kx^n+x^{2n}} = \frac{1}{n}\cdot\frac{1-x^2}{1-x^{2n}}\left\{\frac{1}{1-2ax+x^2} + \frac{1}{1-2bx+x^2} + \ldots\right\}$

But $\frac{1-x^2}{1-x^{2n}} = \frac{x^2-1}{x^{2n}-1} = 1 - \frac{x^{2n}-x^2}{x^{2n}-1} = 1 - \frac{x^{2n-1}-x}{x^{2n}-1}x \ldots$

$= 1 - \frac{x^{n-1}-\frac{1}{x^{n-1}}}{x^n-\frac{1}{x^n}}x = 1 - \frac{\left(x^{n-1}-\frac{1}{x^{n-1}}\right)\left(x^n-\frac{1}{x^n}\right)}{\left(x^n-\frac{1}{x^n}\right)^2}x$

$= 1 - \frac{x^{2n-1}-x-\frac{1}{x}+\frac{1}{x^{2n-1}}}{\left(x^n+\frac{1}{x^n}\right)^2-4}x.$

Let the first term be the one which is required; then, since the factor is $1 - 2ax + x^2 = 0$, we have $x + \frac{1}{x} = 2a$ $= 2\cos.\frac{\theta}{n}$; wherefore $x^{2n-1} + \frac{1}{x^{2n-1}} = 2\cos.(2n-1)\frac{\theta}{n}$; and the required term is $\frac{1}{n}\cdot\frac{1-\frac{2g-2a}{4k^2-4}x}{1-2ax+x^2}$ . . . .

$= \frac{1}{n}\cdot\frac{1-\frac{a-g}{2k^2}x}{1-2ax+x^2}$, where $g = \cos.(2n-1)\frac{\theta}{n}$, and $k = \sin.\theta$.

Similarly, the second term is $\frac{1}{n}\cdot\frac{1-\frac{a-h}{2k^2}x}{1-2bx+x^2}$, where $h = \cos.(2n-1)\frac{2\pi+\theta}{n}$, and so on.

By means of this article, the form $\int \frac{dx}{1-2kx^n+x^{2n}}$ may be integrated and exhibited.

14. *Required to integrate* $\frac{x^n dx}{1-2kx^n+x^{2n}} = du$, *where* $k<1$.

T. Simpson integrates this form in the following manner.

Assume $k = \cos.\theta$; let $a = \cos.\frac{\theta}{n}$, $b = \cos.\frac{2\pi+\theta}{n}$, . . $c = \cos.\frac{4\pi+\theta}{n}$ . . .; then $1-2kx^n+x^{2n} = (1-2ax+x^2)(1-2bx+x^2)$ . . .

Differentiate this on the supposition that θ is the only variable; and let $k' = \sin.\theta$, $a' = \sin.\frac{\theta}{n}$, $b' = \sin.\frac{2\pi+\theta}{n}$, . . . then we have

$$\frac{x^n k'}{1-2kx^n+x^{2n}} = \frac{xa'}{1-2ax+x^2} + \frac{xb'}{1-2bx+x^2} + \ldots$$

or $du = \frac{1}{k'}\left\{\frac{a'xdx}{1-2ax+x^2} + \frac{b'xdx}{1-2bx+x^2} + \ldots\right\}$ each of which is integrable.

By the same method the form $\frac{x^{m+n}dx}{1-2kx^n+x^{2n}}$, may be separated into fractions of the form $\frac{sx^m dx}{1-2ax+x^2}$ each of which is integrable. The student may see these forms exhibited in T. Simpson's Fluxions, vol. ii. pp. 104.

We shall conclude this part of the subject with Cotes' and De Moivre's theorems, which were invented for the purpose of resolving binomials and trinomials into their quadratick divisors.

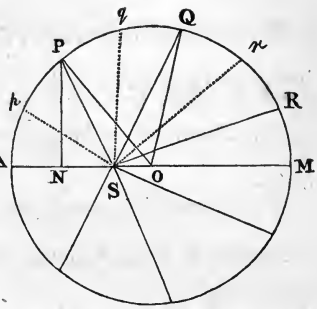
15. *Cotes' Theorem.*

If any point S *be taken in* AO *the radius of a circle whose centre is* O, *and the circumference be divided into n equal parts,* AP, PQ, QR, . . . *then shall* SP × SQ × SR . . . = $AO^n - SO^n$.

Also, if AP, PQ, QR, . . . *be bisected in p, q, r, . . . then shall* sp × sq × sr . . . = $AO^n + SO^n$.

Draw the ordinate PN; join OP, OQ, ... and let

AO $= r$	$a = \cos.\angle$ AOP
SO $= x$	$b = \cos.\angle$ AOQ
	&c. = &c.



(Eu. 2. 13.) $SP^2 = PO^2 + OS^2 - 2SO \times ON = r^2 + x^2 - 2arx$. Similarly, $SQ^2 = r^2 - 2brx + x^2$, $SR^2 = r^2 - 2crx + x^2 \ldots$; wherefore $SP^2 \times SQ^2 \times SR^2 \ldots = (r^2 - 2arx + x^2)(r^2 - 2brx + x^2)(r^2 - 2crx + x^2) \ldots$

But it may be shown as in Art. 6, Cor. 3, that $(r^2 - 2arx + x^2)^{\frac{1}{2}}(r^2 - 2brx + x^2)^{\frac{1}{2}} \ldots = r^n - x^n$; whence $SP \times SQ \times SR \ldots = r^n - x^n = AO^n - SO^n$.

Also, since p, P, q, Q, r, ... divide the circumference into $2n$ equal parts, we have $sp \times SP \times sq \ldots = AO^{2n} - SO^{2n}$; wherefore $sp \times sq \times sr \ldots = \frac{AO^{2n} - SO^{2n}}{AO^n - SO^n} = AO^n + SO^n$.

If s be taken in OA produced, x is greater than r; and it may be shown in the same manner that $SP \times SQ \times SR \ldots = SO^n - AO^n$.

Cor. 1. Since the radii vectores, reckoning from the last but one, are the same as those from the first, we have $AO^n \sim SO^n = SA \times SP^2 \times SQ^2 \ldots$

If n is odd, and ASM be the diameter of the circle, one of the divisions will fall on M, and we have . . . . . $AO^n \sim SO^n = SA \times SM \times SP^2 \times SQ^2 \ldots$

Cor. 2. $AO^n + SO^n = sp^2 \times sq^2 \times sr^2 \ldots$ to $\frac{n}{2}$ terms.

16. *De Moivre's Theorem.*

If the divisions of the circle begin from some other radius OB *than that in which* S *is situated, then shall* $SB^2 \times SP^2 \times SQ^2 \times SR^2 \ldots = AO^{2n} - 2AO^n \times SO^n \cos.\theta + SO^{2n}$, *where* $\angle AOB = \frac{\theta}{n}$.

For as before, $SP^2 = PO^2 + OS^2 - 2SO \times ON = r^2 + x^2 - 2rx \cos.\frac{2\pi + \theta}{n}$; and consequently $SB^2 \times SP^2 \times SQ^2 \times SR^2 \ldots$

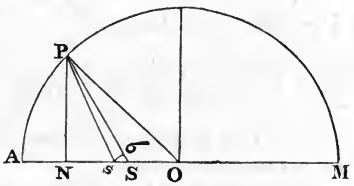
$= (r^2 - 2rx\cos.\frac{\theta}{n} + x^2)(r^2 - 2rx\cos.\frac{2\pi+\theta}{n} + x^2)\ldots$

$= $ (Art. 4.), $r^{2n} - 2rx\cos.\theta + x^{2n} = $ AO^{2n} $-$ 2AO^n $\times$ SO$^n\cos.\theta$ + SO^{2n}.

17. *Required to construct the fluents of* $\frac{dx}{x^2-2ax+1}$ *and of* $\frac{xdx}{x^2-2ax+1}$, *when* $a<1$.

With radius OA describe a semicircle APM.

Let OA $= 1$; suppose $a =$ ON $=$ cos. $\angle$ AOP a constant quantity; and suppose OS to represent the variable quantity x.



Then joining SP, if OS become Os and $s\sigma$ be drawn perpendicular on SP, $d.\angle$ OPS $=$ the limit of $\angle$ SP$s = \frac{s\sigma}{SP}$; but $s\sigma$: Ss :: NP : SP; whence $d.\angle$ OPS $=$ the limit of $\frac{NP \times Ss}{SP^2}$ $= \frac{\text{sin.AP}\,.\,dx}{x^2-2ax+1}$, and, integrating, $\int\frac{dx}{x^2-2ax-1}$ . . . . $= \frac{1}{\text{sin.AP}} \times \angle$OPS.

Also $\frac{xdx}{x^2-2ax+1} = \frac{xdx-adx}{x^2-2ax+1} + \frac{adx}{x^2-2ax+1}$, and, integrating, $\int\frac{xdx}{x^2-2ax+1} = l\sqrt{x^2-2ax+1} + \frac{a}{\text{sin.AP}}\cdot\angle$ OPS.

$$= l\frac{SP}{OP} + \text{cot.AP} \times \angle \text{OPS}.$$

The form $\frac{x^2dx}{x^2-2ax+1}$ by division becomes $dx + \frac{2axdx-dx}{x^2-2ax+1}$, and is therefore integrable; and generally $\frac{x^mdx}{x^2-2ax+1}$ is integrable by this construction, if m is a positive integer.

Let m be a fraction $= \frac{p}{q}$; then substituting $y = x^{\frac{1}{q}}$, or $x = y^q$, we have $\frac{x^{\frac{p}{q}}dx}{x^2 - 2ax + 1} = \frac{qy^{p+q-1}dy}{y^{2q} - 2ay^q + 1}$, which is to be integrated as in the preceding article.

The forms $\frac{x^{-1}dx}{x^2 - 2ax + 1}$, $\frac{x^{-2}dx}{x^2 - 2ax + 1}$, dividing by $1 - 2ax + x^2$, become $\frac{dx}{x} + \frac{2adx - xdx}{x^2 - 2ax + 1}$ and . . . . $\frac{dx}{x^2} + \frac{2ax^{-1}dx - dx}{x^2 - 2ax + 1}$, and are therefore integrable.

If m is a negative fraction, it may be reduced to an integral form as before.

De Moivre (Misc. Anal. pp. 60.) integrates the form $\frac{x^m dx}{(x^{2n} - 2kx^n + 1)^v}$.

18. *Irrational Forms.*

If these can be rendered rational by substitution, or any other artifice of calculation, they are to be integrated as in the preceding articles, and their fluents may be expressed in finite terms: if they cannot be rationalized, their integration, when possible, is to be effected by the method of continuation.

19. It has been shown (vol. i. ch. 2. art. 46.) under what conditions binomial forms may be rationalized by substitution; and it may be here observed, that the more general form $\text{F}\left\{x^{rn}, (a + bx^n)^{\frac{p}{q}}, (a + bx^n)^{\frac{s}{t}}, (a + bx^n)^{\frac{v}{w}}, \ldots\right\} x^{n-1}dx$, where F represents any rational function of the quantities which compose it, may be likewise integrated by the same method.

For substitute $a + bx^n = y^{rqt\ldots}$; then $x^n = \frac{y^{rqt\ldots} - a}{b}$, and the form becomes $\text{F}\left\{\left(\frac{y^{rqt\ldots} - a}{b}\right)^r, y^{rpt\ldots}, y^{rqs\ldots}, \ldots\right\} \frac{rqt\ldots}{nb} y^{rqt\ldots - 1}dy$; which is a rational function of y.

Thus $\dfrac{dx}{(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{3}}}$ may be rationalized by substituting $(1+x)^{\frac{1}{6}} = y$, or $y^6 = 1+x$.

This substitution enables us to effect the integration, even if the binomials which compose the function are fractional, provided that they are, as before, the same binomials.

Thus, $\mathrm{F}\left\{x^m, \left(\dfrac{a+bx^n}{a'+b'x^n}\right)^{\frac{p}{q}}, \left(\dfrac{a+bx^n}{a'+b'x^n}\right)^{\frac{s}{t}}, \ldots\right\} x^{n-1}dx$ is rationalized by substituting $\dfrac{a+bx^n}{a'+b'x^n} = y^{rqt\cdots}$.

The form $(x+\sqrt{1+x^2})^{\frac{p}{q}}\,\mathrm{R}dx$, where R represents a rational function either of x or of $\sqrt{1+x^2}$ may be always rationalized by substituting $x+\sqrt{1+x^2} = y^q$.

If different binomials compose the function, the form can in general be integrated only by particular artifices, of which a few examples have been given in the first volume.

The case in which there are only two different binomials of the form $\sqrt{a+bx}$, $\sqrt{a'+b'x}$ is integrable by substituting $a+bx = (a'+b'x)y^2$; for then we have

$$x = \frac{a-a'y^2}{b'y^2-b},\ \sqrt{a+bx} = \frac{y\sqrt{ab'-a'b}}{\sqrt{b'y^2-b}},\ \sqrt{a'+b'x} \cdots = \frac{\sqrt{ab'-a'b}}{\sqrt{b'y^2-b}};$$

or the proposed form is reducible to one which contains only one binomial surd $\sqrt{b'y^2-b}$; and it may be rationalized by a second substitution.

The form $\dfrac{x^{rn-1}dx}{(a+bx^n)^p(g+hx^n)^q}$ is integrable by substitution (vol. i. ch. 2. art. 50.), if r be a positive integer, and $p+q$ also an integer greater than r.

For substitute $y = \dfrac{a+bx^n}{g+hx^n}$; then $x^n = \dfrac{gy-a}{b-hy}$, $\cdots$

$$\therefore x^m = \left(\frac{gy-a}{b-hy}\right)^r,\ \therefore x^{rn-1}dx = \frac{1}{n}\left(\frac{gy-a}{b-hy}\right)^{r-1}\left\{\frac{gdy}{b-hy} + \frac{(gy-a)hdy}{(b-hy)^2}\right\} = \frac{bg-ah}{n}\cdot\frac{(gy-a)^{r-1}dy}{(b-hy)^{r+1}}.$$

Also $a + bx^n = a + \frac{b(gy-a)}{b-hy} = \frac{(bg-ah)y}{b-hy}$; and

$g + hx^n = g + \frac{h(gy-a)}{b-hy} = \frac{bg-ah}{b-hy}$, $\therefore$ . . . . . .

$du = \frac{bg-ah}{n} \times \frac{(gy-a)^{r-1}dy}{(b-hy)^{r+1}} \div \frac{(bg-ah)^{p+q}y^p}{(b-hy)^{p+q}}$ . . . .

$= \frac{1}{n(bg-ah)^{p+q-1}} \times (gy - a)^{r-1}(b - hy)^{p+q-r-1}y^{-p}dy$;

which can be expressed in finite terms if r be a positive integer, and $p + q$ an integer greater than r *.

20. In the following forms, integrable by the method of continuation, we shall suppose the indices of x both within and without the radical to be integers; for if they are not, they may be made integers by substitution; thus, let the form be $(a + bx^{\frac{1}{2}})^p x^{\frac{1}{3}}dx$; substitute $y^6 = x$, and it becomes $6(a + by^3)^p y^7 dy$. Also the index of x within the radicals may be supposed positive; for if it be negative, it may be made positive by substitution, or it may be brought out of the radical. We shall further suppose, that there is one term of the quantity within the radical into which x does not enter, as it may be reduced to this form.

21. *In a binomial form, the index of the variable without the radical may be increased or diminished by its index within the radical.*

For let it be required to reduce $\int \mathrm{x}^p x^{v+n}dx = \mathrm{B}$ to the form $\int \mathrm{x}^p x^v dx = \mathrm{A}$, where $\mathrm{x} = a + bx^n$.

Assume $\mathrm{P} = \mathrm{x}^{p+1}x^{v+1}$;

then $d\mathrm{P} = (p + 1)nb\mathrm{x}^p x^{v+n}dx + (v + 1)\mathrm{x}^{p+1}x^v dx$,

$= \{(p+1)nb + (v+1)b\}\mathrm{x}^p x^{v+n}dx + (v+1)a\mathrm{x}^p x^v dx$,

or $\mathrm{P} = \{(p+1)n + (v+1)\}b\mathrm{B} + (v+1)a\mathrm{A}$. (α).

From this equation we can find either B in terms of A, or A in terms of B.

If it be required to reduce $\int \mathrm{x}^p x^{v-n}dx$ to the form $\int \mathrm{x}^p x^v dx$; here $d\mathrm{A} = \mathrm{x}^p x^{v-n}dx$, $d\mathrm{B} = \mathrm{x}^p x^v dx$; wherefore assume

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* The reader should consult a very useful paper in the Philosophical Transactions, June 4, 1816, on the Fluents of Irrational Functions by Mr. Bromhead, who by means of inverse functions integrates forms more general than those given by any preceding writer.

$$\text{P} = \text{X}^{p+1}x^{v+1-n};$$

$$\text{and P} = \{(p+1)n+(v+1-n)\}b\text{B}+(v+1-n)a\text{A},$$
$$=(np+v+1)b\text{B}+(v+1-n)a\text{A}. \quad (\beta).$$

Let $v+1=m$ or $v=m-1$; then (α) and (β) become

$$\left.\begin{aligned}\text{X}^{p+1}x^{m} &= (m+n+np)b\int\text{X}^{p}x^{m+n-1}dx + ma\int\text{X}^{p}x^{m-1}dx\\ \text{X}^{p+1}x^{m-n} &= (m+np)b\int\text{X}^{p}x^{m-1}dx + (m-n)a\int\text{X}^{p}x^{m-n-1}dx\end{aligned}\right\}\text{or}$$

$$\left.\begin{aligned}\int\text{X}^{p}x^{m-1}dx &= \frac{\text{X}^{p+1}x^{m}}{ma} - \frac{(m+n+np)b}{ma}\int\text{X}^{p}x^{m+n-1}dx\ (\alpha).\\ \int\text{X}^{p}x^{m-1}dx &= \frac{\text{X}^{p+1}x^{m-n}}{(m+np)b} - \frac{(m-n)a}{(m+np)b}\int\text{X}^{p}x^{m-n-1}dx\ (\beta).\end{aligned}\right\}$$

These two formulæ by which the index of x without the radical is increased or diminished by its index within, are Hirsch's Table I. Formulæ 5 and 3.

Cor. 1. Hence by successive substitutions the index of x without the radical may be increased or diminished by any multiple of n.

Cor. 2. The form (α) fails when $m = 0$ or $v = -1$; in this case $d\text{A} = \dfrac{(a+bx^n)^p dx}{x}$; or representing the exponent of the radical by a fraction, $d\text{A} = \dfrac{(a+bx^n)^{\frac{p}{q}}dx}{x}$.

Substitute $y^q = a+bx^n$; then $(a+bx^n)^{\frac{p}{q}} = y^p$. Also $\cdots$ $bx^n = y^q - a$ or $l.\ bx^n = l(y^q-a)$, $\therefore \dfrac{dx}{x} = \dfrac{q}{n}\ \dfrac{y^{q-1}dy}{y^q-a}$, and $\cdots$ $d\text{A} = \dfrac{q}{n}\ \dfrac{y^{p+q-1}dy}{y^q-a}$, which is under a rational form.

Cor. 3. The form (β) fails when $m+np=0$ or $p=-\dfrac{m}{n}$; in this case $d\text{A} = (a+bx^n)^{-\frac{m}{n}}x^{m-1}dx$; which is rationalized by substituting $a+bx^n = x^n y^n$; when it becomes $-\dfrac{y^{-m+n-1}dy}{y^n-b}$.

If either of the constants a, b be supposed $= 0$, the form becomes a monomial; and there is no difficulty in the integration.

22. EXAMPLES.

Ex. 1. $\int x^{-p}x^{-m-1}dx =$

$$-\frac{x^{-p+1}x^{-m-n}}{(m+np)b} - \frac{(m+n)a}{(m+np)b}\int x^{-p}x^{-m-n-1}dx.$$

For here $A = \int x^{-p}x^{-m-n-1}dx$, $\therefore v+1 = -m-n \ldots$ $\therefore P$, which $= x^{-p+1}x^{-m-n}$, $= ((-p+1)n-m-n)b\int x^{-p}x^{-m-1}dx$ $-(m+n)a\int x^{-p}x^{-m-n-1}dx$, $\therefore$ . . . . . .

$$\int x^{-p}x^{-m-1}dx = -\frac{x^{-p+1}x^{-m-n}}{(m+np)b} - \frac{(m+n)a}{(m+np)b}\int x^{-p}x^{-m-n-1}dx.$$

Ex. 2. Required to reduce $\int \frac{dz}{z^{n+1}\sqrt{bz^n-a}}$ to the form $\int \frac{dz}{z\sqrt{bz^n-a}}$.

Here $dA = \frac{dz}{z^{n+1}\sqrt{bz^n-a}}$, or $v = -n-1$, or $v+1 = -n$, and $p = -\frac{1}{2}$; wherefore assume $P = \sqrt{bz^n-a}.z^{-n}$; then

$$dP = \frac{nb}{2}\frac{dz}{z\sqrt{bz^n-a}} - n\sqrt{bz^n-a}.z^{-n-1}dz = nadA \ldots$$

$$-\frac{nb}{2}\frac{dz}{z\sqrt{bz^n-a}}, \therefore A = \frac{\sqrt{bz^n-a}}{naz^n} + \frac{b}{2a}\int \frac{dz}{z\sqrt{bz^n-a}}.$$

Ex. 3. Required to integrate $\frac{x^{3n-1}dx}{\sqrt{a+bx^n}}$.

Let $dA = \frac{x^{n-1}dx}{\sqrt{a+bx^n}}$ or $A = \frac{2}{nb}\sqrt{a+bx^n}$; . . . .

$dB = \frac{x^{2n-1}dx}{\sqrt{a+bx^n}}$; $dC = \frac{x^{3n-1}dx}{\sqrt{a+bx^n}}$.

First assume $P = \sqrt{a+bx^n}.x^n$, $\therefore dP = \frac{nb}{2}\frac{x^{2n-1}dx}{\sqrt{a+bx^n}} \ldots$

$$+n\sqrt{a+bx^n}.x^{n-1}dx = \frac{3nb}{2}dB + nadA, \therefore B = \frac{2P}{3nb} - \frac{2aA}{3b}.$$

Next assume $Q = \sqrt{a+bx^n}.x^{2n}$, $\therefore dQ = \frac{nb}{2}\frac{x^{3n-1}dx}{\sqrt{a+bx^n}}$

$$+2n\sqrt{a+bx^n}.x^{2n-1}dx = \frac{5nb}{2}dC + 2nadB, \quad . \ . \ . \ . \ .$$

$$\therefore \text{C} = \frac{2\text{Q}}{5nb} - \frac{4a\text{B}}{5b} = \frac{2\text{X}^{\frac{1}{2}}}{5nb}\left\{ x^{2n} - \frac{2.2ax^n}{1.3b} \right\} + \frac{2.4a^2\text{A}}{3.5b^2}.$$

Ex. 4. Required to integrate $\sqrt{a^n - x^n}.x^{\frac{3n}{2}-1}dx = du$.

To reduce this to $\int \sqrt{a^n - x^n}.x^{\frac{n}{2}-1}dx$, assume

$\text{P} = (a^n - x^n)^{\frac{3}{2}}x^{\frac{n}{2}}$; then $d\text{P} = -\frac{3n}{2}du + \frac{n}{2}(a^n - x^n)^{\frac{3}{2}}x^{\frac{n}{2}-1}dx$

$= -2ndu + \frac{na^n}{2}\sqrt{a^n - x^n}.x^{\frac{n}{2}-1}dx$, or . . . . . .

$$u = \frac{a^n}{4}\int \sqrt{a^n - x^n}.x^{\frac{n}{2}-1}dx - \frac{(a^n - x^n)^{\frac{3}{2}}x^{\frac{n}{2}}}{2n}.$$

To find $\int \sqrt{a^n - x^n}.x^{\frac{n}{2}-1}dx = d\text{F}$, substitute $a^n = b^2$, $x^n = y^2$; then $d\text{F} = \frac{2}{n}\sqrt{b^2 - y^2}\,dy$; and $\text{F} = \frac{2}{n}$ cir. area, rad. $= b$, absc. $= y$, $= \frac{2}{n}$ cir. area, rad. $= a^{\frac{n}{2}}$, absc. $= x^{\frac{n}{2}}$, $= \frac{2a^n}{n}$ cir. area, absc. $= \frac{x^{\frac{n}{2}}}{a^{\frac{n}{2}}}$; wherefore . . . . . .

$$u = \frac{a^{2n}}{2n} \text{ cir. area, absc.} = \frac{x^{\frac{n}{2}}}{a^{\frac{n}{2}}}, -\frac{(a^n - x^n)^{\frac{3}{2}}x^{\frac{n}{2}}}{2n}.$$

Ex. 5. $\int \frac{x^{2n}dx}{\sqrt{a + bx^2}}$, where n is a positive integer.

Let $d\text{A} = \frac{dx}{\sqrt{a + bx^2}}$, $d\text{B} = \frac{x^2dx}{\sqrt{a + bx^2}}$, $d\text{C} = \frac{x^4dx}{\sqrt{a + bx^2}}$ . . .

1st, Assume $\text{P} = \text{X}^{\frac{1}{2}}x$, for $v = 0$; then $d\text{P} = bd\text{B} + \text{X}^{\frac{1}{2}}dx$

$= 2bd\text{B} + ad\text{A}$; $\therefore \text{P} = 2b\text{B} + a\text{A}$ or $\text{B} = \frac{\text{P} - a\text{A}}{2b} = \frac{\text{X}^{\frac{1}{2}}x - a\text{A}}{2b}$.

2dly, Assume $\text{Q} = \text{X}^{\frac{1}{2}}x^3$; then $d\text{Q} = bd\text{C} + \text{X}^{\frac{1}{2}}.3x^2dx$ . .

$= 4bd\text{C} + 3ad\text{B}$, $\therefore \text{Q} = 4b\text{C} + 3a\text{B}$ or $\text{C} = \frac{\text{Q} - 3a\text{B}}{4b}$ . . .

$$= \frac{x^{\frac{1}{2}}x^3}{4b} - \frac{3ax^{\frac{1}{2}}x}{4.2b^2} + \frac{3a^2A}{4.2b^2}.$$

3dly, Assume $R = x^{\frac{1}{2}}x^5$; then $dR = 6bdD + 5adc$;

$$\therefore D = \frac{R - 5aC}{6b} = \frac{x^{\frac{1}{2}}x^5}{6b} - \frac{5ax^{\frac{1}{2}}x^3}{6.4b^2} + \frac{5.3a^2x^{\frac{1}{2}}x}{6.4.2b^3} - \frac{5.3a^3A}{6.4.2b^3}.$$

From which the law is sufficiently manifest; viz. that

$$\int \frac{x^{2n}dx}{\sqrt{a+bx^2}} = x^{\frac{1}{2}}\left\{\frac{x^{2n-1}}{2n.b} - \frac{2n-1}{2n(2n-2)}\frac{ax^{2n-3}}{b^2} \cdots \right.$$
$$\left. + \frac{(2n-1)(2n-3)}{2n(2n-2)(2n-4)}\frac{a^2x^{2n-5}}{b^3} - \cdots\right\}$$
$$\pm \frac{(2n-1)(2n-3)\cdots}{2n(2n-2)(2n-4)\cdots}\cdot\frac{a^nA}{b^n}.$$

The coefficient of the last term is + or −, according as n is even or odd.

Similarly, it may be shown that

$$\int \frac{x^{2m+1}dx}{\sqrt{a+bx^2}} = x^{\frac{1}{2}}\left\{\frac{x^{2m}}{(2m+1)b} - \frac{2max^{2m-2}}{(2m+1)(2m-1)b^2} \cdots \right.$$
$$\left. + \frac{2m(2m-2)a^2x^{2m-4}}{(2m+1)(2m-1)(2m-3)b^3} - \cdots\right\}$$

This latter form may also be integrated by substitution as in vol. i. c. ii. art. 46; and in either case the fluent does not contain a circular arc or a logarithm.

Similarly may be exhibited the forms $\int \sqrt{a+bx^2}\, x^{2n}dx$ and $\int \sqrt{a+bx^2}\, .\, x^{2m+1}dx$ where b may be either positive or negative.

Ex. 6. $du = \dfrac{x^{-2}dx}{\sqrt{a-x}}$.

Assume $P = x^{-1}\sqrt{a-x}$; then $dP = -x^{-2}\sqrt{a-x}\,dx - \frac{1}{2}\dfrac{x^{-1}dx}{\sqrt{a-x}} = -adu + \frac{1}{2}\dfrac{x^{-1}dx}{\sqrt{a-x}}$.

Since $\dfrac{x^{-1}dx}{\sqrt{a-x}}$ cannot be further reduced, for $v + 1 = 0$, substitute $y = \sqrt{a-x}$; then $\dfrac{x^{-1}dx}{\sqrt{a-x}} = \dfrac{-2dy}{a-y^2}$; $\therefore dP = -adu - \dfrac{dy}{a-y^2}$, and $P = -au - \dfrac{1}{2a^{\frac{1}{2}}}\, l\, \dfrac{a^{\frac{1}{2}}+y}{a^{\frac{1}{2}}-y}$, ... or

$$u = -\frac{\text{P}}{a} - \frac{1}{2a^{\frac{3}{2}}}\, l\, \frac{a^{\frac{1}{2}} + \sqrt{a-x}}{a^{\frac{1}{2}} - \sqrt{a-x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$= -\frac{\sqrt{a-x}}{ax} - \frac{1}{a^{\frac{3}{2}}}\, l\, \frac{a^{\frac{1}{2}} + \sqrt{a-x}}{x}.$$

23. *Required to exhibit* $\int \text{X}^p x^{m+rn-1} dx$ *in terms of* $\int \text{X}^p x^{m-1} dx$.

Let $d\text{A} = \text{X}^p x^{m-1} dx$; $d\text{B} = \text{X}^p x^{m+n-1} dx$; $d\text{C} = \text{X}^p x^{m+2n-1} dx$; . . . $d\text{K} = \text{X}^p x^{m+(r-1)n-1} dx$; and $d\text{L} = \text{X}^p x^{m+rn-1} dx$.

1st, Assume $\text{P} = \text{X}^{p+1} x^m$; then $d\text{P} = (p+1)nb\text{X}^p x^{m+n-1} dx + m\text{X}^{p+1} x^{m-1} dx = ((p+1)n + m)bd\text{B} + mad\text{A}$, $\therefore$ . . . $\text{A} = \dfrac{\text{P} - (np+n+m)b\text{B}}{ma}$.

2dly, Assume $\text{Q} = \text{X}^{p+1} x^{m+n}$; then $d\text{Q} = ((p+1)n+m+n)bd\text{C} + (m+n)ad\text{B}$, $\therefore$ $\text{B} = \dfrac{\text{Q} - (np+2n+m)b\text{C}}{(m+n)a}$.

3dly, Assume $\text{R} = \text{X}^{p+1} x^{m+2n}$; then

$$\text{C} = \frac{\text{R} - (np+3n+m)b\text{D}}{(m+2n)a}.$$

Similarly, $\text{K} = \dfrac{\text{X}^{p+1} x^{m+(r-1)n} - (np+rn+m)b\text{L}}{(m+(r-1)n)a}$; wherefore $\int \text{X}^p x^{m-1} dx =$ . . . . . .

$$\text{X}^{p+1} x^m \left\{ \text{A}' - \text{B}' x^n + \text{C}' x^{2n} \cdots \pm \text{L}' x^{(r-1)n} \right\} \quad . \quad . \quad . \quad . \quad .$$

$$\mp \text{L}'(np + rn + p)b \int \text{X}^p x^{m+rn-1} dx,$$

where the values of A′, B′, C′, . . . &c. are given by the following equations:

$$\text{A}' = \frac{1}{ma};\ \text{B}' = \frac{np+n+m}{(m+n)a}\text{A}';\ \text{C}' = \frac{np+2n+m}{(m+2n)a}\text{B}';\ .\ .\ .\ .\ .$$

$$\text{L}' = \frac{np+(r-1)n+m}{(m+(r-1)n)a}\text{K}.$$

24. *Required to exhibit* $\int \text{X}^p x^{m-rn-1} dx$ *in terms of* $\int \text{X}^p x^{m-1} dx$.

Let $d\text{A} = \text{X}^p x^{m-1} dx$, $d\text{B} = \text{X}^p x^{m-n-1} dx$, $d\text{C} = \text{X}^p x^{m-2n-1} dx$... $d\text{K} = \text{X}^p x^{m-(r-1)n-1} dx$ and $d\text{L} = \text{X}^p x^{m-rn-1} dx$.

1st, Assume $\text{P} = \text{X}^{p+1} x^{m-n}$; then $d\text{P} = (p+1)nb\text{X}^p x^{m-1} dx + (m-n)\text{X}^{p+1} x^{m-n-1} dx = (np+m)bd\text{A} + (m-n)ad\text{B}$, . . $\therefore$ $\text{A} = \dfrac{\text{P} - (m-n)a\text{B}}{(np+m)b}$.

2dly, Assume $\text{Q} = \text{X}^{p+1}x^{m-2n}$; then
$d\text{Q} = (p+1)nb\text{X}^{p}x^{m-n-1}dx + (m-2n)\text{X}^{p+1}x^{m-2n-1}dx \ldots$
$= (np-n+m)bd\text{B} + (m-2n)adc$, $\therefore \text{B} = \dfrac{\text{Q}-(m-2n)ac}{(np-n+m)b}$.

Similarly, $\text{C} = \dfrac{\text{R}-(m-3n)a\text{D}}{(np-2n+m)b}$ . . . . . and

$\text{K} = \dfrac{\text{X}^{p+1}x^{m-rn} - (m-rn)a\text{L}}{(np-(r-1)n+m)b}$; wherefore $\int x^{p}x^{m-1}dx \ldots$
$= \text{X}^{p+1}\{\text{A}x^{m-n} - \text{B}x^{m-2n} + \text{C}x^{m-3n} - \ldots \pm \text{L}x^{m-rn}\} \ldots$
$\mp (m-rn)a\text{I}\int x^{p}x^{m-rn-1}dx$, where $\text{A} = \dfrac{1}{(np+m)b}$; . . . .

$\text{B} = \dfrac{(m-n)a}{(np-n+m)b}\text{A}$; $\text{C} = \dfrac{(m-2n)a}{(np-2n+m)b}\text{B}$; . . . . .

$\text{L} = \dfrac{(m-(r-1)n)a}{(np-(r-1)n+m)b}\text{K}$.

This and the preceding are Hirsch's Table I. Formulæ 7 and 9.

Cor. When m is a positive multiple of n, the series will terminate, and the fluent can be expressed in finite terms; which agrees with vol. i. c. ii. art. 46.

25. EXAMPLES.

Ex. 1. $du = \dfrac{x^{\frac{5n}{2}-1}dx}{\sqrt{a+bx^{n}}}$.

Substitute $y^2 = x^n$; then $du = \dfrac{2}{n}\cdot\dfrac{y^4dy}{\sqrt{a+by^2}}$.

To integrate $\dfrac{y^4dy}{\sqrt{a+by^2}} = dc$, let $\dfrac{y^2dy}{\sqrt{a+by^2}} = d\text{B}$, and

$\dfrac{dy}{\sqrt{a+by^2}} = d\text{A}$.

Assume $\text{P} = y\sqrt{a+by^2}$; then $d\text{P} = dy\sqrt{a+by^2} \ldots$
$+ \dfrac{by^2dy}{\sqrt{a+by^2}} = ad\text{A} + 2bd\text{B}$; or $\text{B} = \dfrac{\text{P}-a\text{A}}{2b}$.

Assume $\text{Q} = y^3\sqrt{a+by^2}$; then $d\text{Q} = 3ad\text{B} + 4bd\text{C}$; or

$$\text{C} = \frac{\text{Q}-3a\text{B}}{4b} = \frac{\text{Q}}{4b} - \frac{3a\text{P}}{8b^2} + \frac{3a^2\text{A}}{8b^2}, \therefore \ldots$$

$$u = \frac{2\sqrt{a+by^2}}{n}\left\{\frac{y^3}{4b} - \frac{3ay}{8b^2}\right\} + \frac{6a^2}{8nb^{\frac{5}{2}}}l.\,(b^{\frac{1}{2}}y + \sqrt{a+by^2})$$

$$= \frac{\sqrt{a+bx^n}}{n}\left\{\frac{x^{\frac{3n}{2}}}{2b} - \frac{3ax^{\frac{n}{2}}}{4b^2}\right\} + \frac{3a^2}{4nb^{\frac{5}{2}}}l(b^{\frac{1}{2}}x^{\frac{n}{2}} + \sqrt{a+bx^n}).$$

Ex. 2. $\int \frac{dz}{z^{rn+1}\sqrt{bz^n-a}}$ in terms of $\int \frac{dz}{z\sqrt{bz^n-a}}$.

Let $d\text{A} = \frac{dz}{z\sqrt{bz^n-a}}$; $d\text{B} = \frac{dz}{z^{n+1}\sqrt{bz^n-a}}$; $\ldots$

$d\text{C} = \frac{dz}{z^{2n+1}\sqrt{bz^n-a}} \ldots d\text{K} = \frac{dz}{z^{(r-1)n-1}\sqrt{bz^n-a}}$, and

$d\text{L} = \frac{dz}{z^{rn-1}\sqrt{bz^n-a}}$.

1st, Assume $\text{P} = \sqrt{bz^n-a}\,.\,z^{-n}$; then

$$d\text{P} = \frac{nb}{2}\frac{dz}{z\sqrt{bz^n-a}} - n\sqrt{bz^n-a}\,.\,z^{-n-1}dz \quad \ldots$$

$$= -\frac{nb}{2}d\text{A} + nad\text{B}, \therefore \text{A} = \frac{-2\text{P}+2na\text{B}}{nb}.$$

2dly, Assume $\text{Q} = \sqrt{bz^n-a}\,.\,z^{-2n}$;

then $d\text{Q} = -\frac{3nb}{2}d\text{B} + 2nad\text{C}, \therefore \text{B} = \frac{-2\text{Q}+2.2na\text{C}}{3nb}$.

Similarly,

$$\text{C} = \frac{-2\text{R}+2.3na\text{D}}{5nb}, \text{ and } \text{K} = \frac{-2\sqrt{bz^n-a}.z^{-rn}+2.rna\text{L}}{(2r-1)nb};$$

wherefore $\int \frac{dz}{z\sqrt{bz^n-a}} = \ldots$

$$\sqrt{bz^n-a}\{\text{A}z^{-n} + \text{B}z^{-2n} + \text{C}z^{-3n} \ldots + \text{L}z^{-rn}\} + \ldots$$

$$rna\text{L}\int \frac{dz}{z^{rn+1}\sqrt{bz^n-a}}; \text{ where } \text{A} = -\frac{2}{nb}; \text{B} = \frac{2a}{3b}\text{A}; \quad \ldots$$

$$\text{C} = \frac{4a}{5b}\text{B}; \ldots \text{L} = \frac{2(r-1)a}{(2r-1)b}\text{K}.$$

The form $\frac{dz}{z\sqrt{bz^n-a}}$ may be integrated by substituting $y = \sqrt{bz^n - a}$ (Vol. i. c. ii. 47, Praxis, Ex. 8.)

Ex. 3. Required the whole fluent of $\frac{x^n dx}{\sqrt{a^2 - x^2}}$ between the values $x = 0$, $x = a$, when n is an even number.

Let $d\text{A} = \frac{dx}{\sqrt{a^2-x^2}}$, $d\text{B} = \frac{x^2dx}{\sqrt{a^2-x^2}}$, $d\text{C} = \frac{x^4dx}{\sqrt{a^2-x^2}}$ . . . $d\text{K} = \frac{x^{n-2}dx}{\sqrt{a^2-x^2}}$; $d\text{L} = \frac{x^n dx}{\sqrt{a^2-x^2}}$.

Assume $\text{P} = x\sqrt{a^2 - x^2}$; then $d\text{P} = a^2 d\text{A} - 2d\text{B}$, or $\text{A} = \frac{\text{P}+2\text{B}}{a^2}$.

Assume $\text{Q} = x^3\sqrt{a^2 - x^2}$; then $d\text{Q} = 3a^2 d\text{B} - 4d\text{C}$, or $\text{B} = \frac{\text{Q}+4\text{C}}{3a^2}$.

. . . . = ...

Assume $\text{T} = x^{n-1}\sqrt{a^2 - x^2}$; then $d\text{T} = (n-1)a^2 d\text{K} - nd\text{L}$, or $\text{K} = \frac{\text{T}+n\text{L}}{(n-1)a^2}$.

But P, Q, . . . T, each $= 0$, both when $x = 0$ and when $x = a$; whence we have

$$(\text{A}) = \frac{2}{a^2} \cdot \frac{4}{3a^2} \cdot \frac{6}{5a^2} \cdots \frac{n-2}{(n-3)a^2} \cdot \frac{n}{(n-1)a^2} (\text{L}),$$

$$\text{or } (\text{L}) = \frac{1.3 \ldots (n-3)(n-1)}{2.4 \ldots (n-2)n} \cdot \frac{\pi a^n}{2}.$$

If n is odd, and $= 2m + 1$, it may be shown in the same manner that $(\text{L}) = \frac{2.4 \ldots (n-3)(n-1)}{3.5 \ldots (n-2)n} a^{2m} (\text{A})$, where $d\text{A} = \frac{xdx}{\sqrt{a^2-x^2}}$; whence $(\text{L}) = \frac{2.4 \ldots (n-3)(n-1)}{3.5 \ldots (n-2)n} a^n$.

Cor. 1. Hence may be deduced Wallis' expression for the circumference of a circle.

For suppose n infinite; then since $2m + 1 = 2m$, we have

$$\int \frac{x^{2m}dx}{\sqrt{a^2-x^2}} = \int \frac{x^{2m+1}dx}{\sqrt{a^2-x^2}} \text{ or } \frac{1.3.5\ldots\text{ad inf.}}{2.4.6\ldots\text{ad inf.}} \cdot \frac{\pi a^n}{2} = \ldots.$$

$\frac{2.4.6\ldots\text{ad inf.}}{1.3.5\ldots\text{ad inf.}} a^n$, or $2\pi = 4 \times \frac{2^2.4^2.6^2\ldots\text{ad inf.}}{1.3^2.5^2\ldots\text{ad inf.}}$, an expression due to Wallis.

Cor. 2. When n is odd, the fluent may be expressed algebraically; when n is even, it is expressed in terms of π.

Cor. 3. $\left(\int \frac{x^{2m}dv}{\sqrt{a^2-x^2}} . \int \frac{x^{2m+1}dx}{\sqrt{a^2-x^2}}\right) = \frac{a^{4m+1}}{2m+1} \cdot \frac{\pi}{2}$.

This obtains even whem m is not an integer.

Cor. 4. Substitute $x = z^r$; then $z = 0$, $z = 1$, when $x = 0$, $x = 1$, and we have

$$r^2\left(\int \frac{z^{r(2m+1)-1}dz}{\sqrt{1-z^{2r}}} . \int \frac{z^{r(2m+2)-1}dz}{\sqrt{1-z^{2r}}}\right) = \frac{1}{2m+1} \cdot \frac{\pi}{2}.$$

Let $v = r(2m+1)-1$; and there results

$$\left(\int \frac{z^v dz}{\sqrt{1-z^2}} . \int \frac{z^{v+r}dz}{\sqrt{1-z^{2r}}}\right) = \frac{1}{r(v+1)} \cdot \frac{\pi}{2}.$$

By means of this formula even elliptick and other transcendental forms may be expressed in terms of π. Thus, let $r = 2$, $v = 0$; and we have

$$\left(\int \frac{dz}{\sqrt{1-z^4}} . \int \frac{z^2dz}{\sqrt{1-z^4}}\right) = \frac{\pi}{4}.$$

Ex. 4. $\frac{x^{\frac{1}{2}}dx}{\sqrt{r^3-x^3}} = du.$

If the index of x without the radical can be diminished by its index within, the form is reduced to

$\frac{x^{-\frac{5}{2}}dx}{\sqrt{r^3-x^3}}$, which $= \frac{x^{-4}dx}{\sqrt{r^3x^{-3}-1}}$, which is integrable.

To effect this reduction, assume $\text{P} = x^{-\frac{5}{2}+1}\sqrt{r^3-x^3}$

$= x^{-\frac{3}{2}}\sqrt{r^3-x^3}$; then $d\text{P} = -\frac{3}{2}x^{-\frac{5}{2}}dx\sqrt{r^3-x^3} - \frac{3}{2}\frac{x^{\frac{1}{2}}dx}{\sqrt{r^3-x^3}}$

$= -\frac{3r^3}{2} \cdot \frac{x^{-\frac{5}{2}}dx}{\sqrt{r^3-x^3}}$, or the method fails; and the reason is that in this example $m + np = 0$. The form may be

integrated by substituting $y^2 = \frac{x^3}{r^3}$; and we find

$u = \frac{2}{3}\sin.^{-1}\frac{x^{\frac{3}{2}}}{r^{\frac{3}{2}}}$. Or by art. 21, cor. 2.

Ex. 5. Required $\int \frac{dx}{x^3(a+bx^2)^2}$ in terms of $\int \frac{dx}{x(a+bx^2)^2}$.

Here by formula (β) art. 21, since $v = -1$, assuming $\text{P} = x^{-2}(a + bx^2)^{-1}$, we have

$d\text{P} = -\frac{2dx}{x^3(a+bx^2)} - \frac{2bdx}{x(a+bx^2)^2} = -2ad\text{A} - 4bd\text{B}$, $\therefore$

$\text{A} = -\frac{2b\text{B}}{a} - \frac{\text{P}}{2a}$; or

$$\int \frac{dx}{x^3(a+bx^2)^2} = -\frac{2b}{a}\int \frac{dx}{x(a+bx^2)^2} - \frac{1}{2ax^2(a+bx^2)}.$$

Ex. 6. $du = \frac{dx}{(1-x^3)^{\frac{1}{3}}}$.

Here $m + np = 0$, so that we cannot reduce du to $\frac{x^{-3}dx}{(1-x^3)^{\frac{1}{3}}} = (x^{-3}-1)^{-\frac{1}{3}}x^{-4}dx$; wherefore (art. 21, cor. 2), substitute $1 - x^3 = x^3y^3$, or $y^3 = \frac{1}{x^3} - 1$, $\therefore x = \frac{1}{(y^3+1)^{\frac{1}{3}}}$ and $dx = -\frac{y^2dy}{(y^3+1)^{\frac{4}{3}}}$.

Also $1 - x^3 = 1 - \frac{1}{y^3+1} = \frac{y^3}{y^3+1}$, $\therefore du = -\frac{ydy}{y^3+1}$, a rational form.

Ex. 7. $du = \frac{x^m dx}{\sqrt{2cx-x^2}}$ where m is a positive integer.

$du = (2c - x)^{-\frac{1}{2}}x^{m-\frac{1}{2}}dx$.

Let $d\text{A} = (2c - x)^{-\frac{1}{2}}x^{-\frac{1}{2}}dx = \frac{dx}{\sqrt{2cx-x^2}}$; . . .

$d\text{B} = (2c - x)^{-\frac{1}{2}}x^{\frac{1}{2}}dx$; $d\text{C} = (2c - x)^{-\frac{1}{2}}x^{\frac{3}{2}}dx$; . . .

$dK = (2c - x)^{-\frac{1}{2}}x^{m-\frac{3}{2}}dx$; $dL = (2c - x)^{-\frac{1}{2}}x^{m-\frac{1}{2}}dx$.

Assume $P = (2c-x)^{\frac{1}{2}}x^{\frac{1}{2}}$; then $dP = -\frac{1}{2}(2c - x)^{-\frac{1}{2}}x^{\frac{1}{2}}dx$ $+ \frac{1}{2}(2c - x)^{\frac{1}{2}}x^{-\frac{1}{2}}dx = cdA - dB$, or $A = \frac{P+B}{c}$.

Assume $Q = (2c - x)^{\frac{1}{2}}x^{\frac{3}{2}}$;

then $dQ = -\frac{1}{2}(2c - x)^{-\frac{1}{2}}x^{\frac{3}{2}}dx + \frac{3}{2}(2c - x)^{\frac{1}{2}}x^{\frac{1}{2}}dx \ldots$

$= 3cdB - 2dC$, or $B = \frac{Q+2C}{3c}$. Similarly $C = \frac{R+3D}{5c}$.

Assume $T = (2c - x)^{\frac{1}{2}}x^{m-\frac{1}{2}}$; then

$dT = -\frac{1}{2}(2c-x)^{-\frac{1}{2}}x^{m-\frac{1}{2}}dx + \frac{2m-1}{2}(2c-x)^{\frac{1}{2}}x^{m-\frac{3}{2}}dx \ldots$

$= (2m - 1)cdK - mdL$, or $K = \frac{T+mL}{(2m-1)c}$; wherefore

$$A = \frac{P}{c} + \frac{Q}{1.3c^2} + \frac{1.2R}{1.3.5c^3} + \cdots \frac{1.2.3\ldots(m-1)T}{1.3.5\ldots(2m-1)c^m} + \ldots$$

$$\frac{1.2.3\ldots m}{1.3.5\ldots(2m-1)c^m}L = \sqrt{2cx-x^2}\left\{\frac{1}{c} + \frac{x}{1.3c^2} + \frac{1.2x^2}{1.3.5c^3}\cdots \right.$$

$$\left. \frac{1.2.3\ldots m-1}{1.3.5\ldots 2m-1}\ \frac{x^{m-1}}{c^m}\right\} + \frac{1.2.3\ldots m}{1.3.5\ldots(2m-1)c^m}L, \quad \ldots \text{ or}$$

$$L = -\sqrt{2cx-x^2}\left\{\frac{x^{m-1}}{m} + \frac{2m-1}{m(m-1)}cx^{m-2} \quad \ldots \ldots \right.$$

$$\left. + \frac{(2m-1)(2m-3)}{m.(m-1)(m-2)}c^2x^{m-3}\ldots + \frac{(2m-1)2m-3)\ldots 5.3.1}{m.(m-1)(m-2)\ldots 3.2.1}c^{m-1}\right\}$$

$$+ \frac{(2m-1)(2m-3)\cdots 5.3.1.c^m}{m.(m-1)(m-2)\cdots 3.2.1}vs^{-1}\frac{x}{c}.$$

26. *In a binomial form the index of the binomial may be increased or diminished by any integer.*

Let it be required to reduce $\int x^{p+r}x^v dx$ where r is an integer to the form $\int x^p x^v dx$ when $x = a + bx^n$.

1st, Let $dA = x^p x^v dx$, $dB = x^{p+1}x^n dx$; then

$B = \int x^{p+1}x^v dx = \int ax^p x^n dx + \int bx^p x^{v+n}dx = aA + \int bx^p x^{v+n}dx$.

But $\int x^p x^{v+n}dx$ may be reduced, as in art. 21, to the form A by assuming $P = x^{p+1}x^{v+1}$; by which we find B in terms of A.

2dly, Let $dC = x^{p+2}x^v dx$; then $C = aB + b\int x^{p+1}x^{v+n}dx$.

But $\int x^{p+1}x^{v+n}dx$ may be reduced to the form B by assuming $Q = x^{p+2}x^{v+1}$; by which we find C in terms of B and A, or in terms of A alone.

Similarly, $D = \int x^{p+3}x^{v}dx$ may be found in terms of C, B, and A, or in terms of A alone.

The index of the binomial has been *diminished* by unity at each assumption; but as the same equation gives A in terms of B, this method enables us to increase the index by any integer.

27. Required to exhibit $\int x^{p+r}x^{m-1}dx$ *in terms of*... $\int x^{p}x^{m-1}dx$.

Let $dA = x^{p}x^{m-1}dx$, $dB = x^{p+1}x^{m-1}dx$, $dC = x^{p+2}x^{m-1}dx$... $dK = x^{p+r-1}x^{m-1}dx$, $dL = x^{p+r}x^{m-1}dx$.

1st, $dB = x\,dA$, $\therefore B = aA + \int bx^{p}x^{m+n-1}dx$.

Assume $P = x^{p+1}x^{m}$;

then $dP = nb(p+1)x^{p}x^{m+n-1}dx + mx^{p+1}x^{m-1}dx$

$= (np + n + m)bx^{p}x^{m+n-1}dx + mad A$;

wherefore $\int bx^{p}x^{m+n-1}dx$

$= \dfrac{x^{p+1}x^{m} - maA}{np+n+m}$, $\therefore B = \dfrac{np+n}{np+n+m}aA + \dfrac{x^{p+1}x^{m}}{np+n+m}$;

or $A = -\dfrac{x^{p+1}x^{m}}{(p+1)na} + \dfrac{np+n+m}{(p+1)na}B$.

2dly, $dC = x\,dB$, $\therefore C = aB + \int bx^{p+1}x^{m+n-1}dx$.

Assume $Q = x^{p+2}x^{m}$;

then $dQ = nb(p+2)x^{p+1}x^{m+n-1}dx + mx^{p+2}x^{m-1}dx = \ldots$

$(np + 2n + m)bx^{p+1}x^{m+n-1}dx + mad B$, ...

or $B = -\dfrac{x^{p+2}x^{m}}{(p+2)na} + \dfrac{np+2n+m}{(p+2)na}C$.

Similarly, $C = -\dfrac{x^{p+3}x^{m}}{(p+3)na} + \dfrac{np+3n+m}{(p+3)na}D$ . . . and

$K = -\dfrac{x^{p+r}x^{m}}{(p+r)na} + \dfrac{np+rn+m}{(p+r)na}L$; wherefore we have

$\int x^{p}x^{m-1}dx = -x^{m}\{AX^{p+1} + BX^{p+2} + CX^{p+2}\ldots + LX^{p+r}\}\ldots$

$+ (np + rn + m)L\int x^{p+r}x^{m-1}dx$; where . . . . . .

$A = \dfrac{1}{(p+1)na}$; $B = \dfrac{np+n+m}{(p+2)na}A$; $C = \dfrac{np+2n+m}{(p+3)na}B$; . . .

$$\text{L} = \frac{np+(r-1)n+m}{(p+r)na}\text{K}.$$

Cor. If either a or b is negative, and x be so assumed that $\text{x} = 0$, we have $(\int \text{x}^{p+r}x^{m-1}dx)$ . . . . . .

$$= \frac{(p+1)(p+2)(p+3)\ldots(p+r)}{(np+n+m)(np+2n+m)\ldots(np+rn+m)}n^r a^r(\int \text{x}^p x^{m-1}dx).$$

28. *Required to exhibit* $\int \text{x}^{p-r}x^{m-1}dx$ *in terms of* . . . $\int \text{x}^p x^{m-1}dx$.

Let $d\text{A} = \text{x}^p x^{m-1}dx$, $d\text{B} = \text{x}^{p-1}x^{m-1}dx$, $d\text{c} = \text{x}^{p-2}x^{m-1}dx$ . . . $d\text{K} = \text{x}^{p-r+1}x^{m-1}dx$, $d\text{L} = \text{x}^{p-r}x^{m-1}dx$; then it may be shown by the same method that

$$\int \text{x}^p x^{m-1}dx = x^m\left\{\text{Ax}^p + \text{Bx}^{p-1} + \text{cx}^{p-2} \ldots + \text{Lx}^{p-r+1}\right\} +$$

$(p-r+1)na\text{I}\int \text{x}^{p-r}x^{m-1}dx$; where $\text{A} = \dfrac{1}{m+np}$, . . .

$$\text{B} = \frac{np}{np-n+m}a\text{A}, \quad \text{c} = \frac{n(p-1)}{np-2n+m}a\text{B}, \quad \ldots \ldots \ldots$$

$$\text{L} = \frac{n(p-r+2)}{np-(r-1)n+m}a\text{K}.$$

These are Hirsch's Table I., Formulæ 10 and 8.

EXAMPLES.

Ex. 1. Required $\int \dfrac{x^m dx}{(1\pm x^n)^2}$ in terms of $\int \dfrac{x^m dx}{1\pm x^n}$.

Let $d\text{A} = \dfrac{x^m dx}{(1\pm x^n)^2}$, $d\text{B} = \dfrac{x^m dx}{1\pm x^n}$; then . . . . . .

$$d\text{B} = d\text{A} \pm \frac{x^{m+n}dx}{(1\pm x^n)^2}, \text{ or } \text{A} = \text{B} \mp \int \frac{x^{m+n}dx}{(1\pm x^n)^2}.$$

To reduce $\dfrac{x^{m+n}dx}{(1\pm x^n)^2} = d\text{F}$, assume $\text{P} = \dfrac{x^{m+1}}{1\pm x^n}$; then . . .

$$d\text{P} = (m+1)d\text{B} \mp n d\text{F}, \text{ or } \pm \text{F} = \frac{m+1}{n}\text{B} - \frac{\text{P}}{n}; \text{ whence}$$

by substitution there results $\text{A} = \dfrac{\text{P}}{n} - \dfrac{m-n+1}{n}\text{B}$; or . . .

$$\int \frac{x^m dx}{(1\pm x^n)^2} = \frac{x^{m+1}}{n(1\pm x^n)} - \frac{m-n+1}{n}\int \frac{x^m dx}{1\pm x^n}. \quad \ldots \ldots$$

(Misc. Analyt. p. 55.)

Ex. 2. $du = \frac{(a+bx^n)^{\frac{5}{2}}dx}{x}$.

Substitute $x^{\frac{n}{2}} = y$; then $\frac{n}{2}lx = ly$ and $\frac{dx}{x} = \frac{2dy}{ny}$, whence

$du = \frac{2}{n}(a+by^2)^{\frac{5}{2}}\frac{dy}{y}$.

To reduce this, let $d\text{A} = \sqrt{a+by^2}\frac{dy}{y}$, ...

$d\text{B} = (a+by^2)^{\frac{3}{2}}\frac{dy}{y}$, $d\text{C} = (a+by^2)^{\frac{5}{2}}\frac{dy}{y}$; then . . . .

$\text{B} = a\text{A} + bf\sqrt{a+by^2}\,ydy = a\text{A} + \frac{(a+by^2)^{\frac{3}{2}}}{3}$, ...

$\text{C} = a\text{B} + bf(a+by^2)^{\frac{3}{2}}ydy = a\text{B} + \frac{(a+by^2)^{\frac{5}{2}}}{5}$ . . . .

$= a^2\text{A} + \frac{a(a+by^2)^{\frac{3}{2}}}{3} + \frac{(a+by^2)^{\frac{5}{2}}}{5}$.

But $\text{A} = af\frac{dy}{y\sqrt{a+by^2}} + bf\frac{ydy}{\sqrt{a+by^2}}$ . . . . . .

$= \sqrt{a}\,l.\frac{\sqrt{a+by^2}-\sqrt{a}}{b^{\frac{1}{2}}y} + \sqrt{a+by^2}$; whence ...

$\text{C or } \frac{nu}{2} = a^{\frac{5}{2}}\,l\,\frac{\sqrt{a+bx^n}-\sqrt{a}}{b^{\frac{1}{2}}x^{\frac{n}{2}}} +$ . . . . . .

$\frac{\sqrt{a+bx^n}}{5}\left\{\frac{23a^2}{3} + \frac{11abx^n}{3} + b^2x^{2n}\right\}$.

Similarly it may be shown that $f\frac{dx}{x(a+bx^n)^{\frac{5}{2}}}$ . . . . .

$= \frac{1}{a^{\frac{5}{2}}}\,l\,\frac{\sqrt{a+bx^n}-\sqrt{a}}{b^{\frac{1}{2}}x^{\frac{n}{2}}} + \frac{4a+3bx^n}{3a^2(a+bx^n)^{\frac{3}{2}}}$.

29. *In a binomial form the index of the binomial may be increased or diminished by any integer; and the index of the variable without the radical may be increased or diminished by any multiple of its index within.*

Let it be required to reduce $\int x^{p+r}x^{v+tn}dx = \text{F}$ to the form $\int x^p x^r dx = \text{A}$, where r and t are integers.

First reduce F to the form $\int x^p x^{v+tn}dx = \text{B}$ by art. 26; and then reduce B to the form A by art. 21, cor. 1; by which F is obtained in terms of A.

Thus $\int x^p x^{m+vn-1}dx = -x^{m+vn}\left\{\text{A}x^{p+1} + \text{B}x^{p+2}\cdots + \text{L}x^{p+r}\right\}$ $+ (n(p+r+v)+m)\text{L}\int x^{p+r}x^{n+vn-1}dx$; where . . . .

$\text{A} = \dfrac{1}{(p+1)na}$; $\text{B} = \dfrac{n(p+1+v)+m}{(p+2)na}\text{A}$; $\text{C} = \dfrac{n(p+2+v)+m}{(p+3)na}\text{B}$..

$\text{L} = \dfrac{n(p+r-1+v)+m}{(p+r)na}\text{K}$ (art. 27).

Also $\int x^p x^{m-1}dx = x^{p+1}x^m\left\{\text{A} - \text{B}x^n + \text{C}x^{2n}\cdots \pm \text{L}x^{m+(v-1)n}\right\}$ $\mp \text{L}(n(p+v)+p)b\int x^p x^{m+vn-1}dx$; where . . . . . .

$\text{A} = \dfrac{1}{ma}$; $\text{B} = \dfrac{np+n+m}{(m+n)a}\text{A}$; $\text{C} = \dfrac{np+2n+m}{(m+2n)a}\text{B}$; . . . . . .

$\text{L} = \dfrac{np+(v-1)n+m}{(m+(v-1)n)a}\text{K}$ (art. 23); and by means of these $\int x^{p+r}x^{m+vn-1}dx$ may be obtained in terms of $\int x^p x^{m-1}dx$.

For the exhibition of this and other general forms, the reader may consult T. Simpson's Fluxions, vol. ii. pp. 46.

Ex. 1. $du = \dfrac{\sqrt{a^2+x^2}dx}{x^3} = \dfrac{a^2dx}{x^3\sqrt{a^2+x^2}} + \dfrac{dx}{x\sqrt{a^2+x^2}}$.

To reduce $\dfrac{dx}{x^3\sqrt{a^2+x^2}} = d\text{F}$, assume $\text{P} = x^{-2}\sqrt{a^2+x^2}$;

then $d\text{P} = -\dfrac{2\sqrt{a^2+x^2}dx}{x^3} + \dfrac{dx}{x\sqrt{a^2+x^2}}$, which shows that

$d\text{P} = -2du + \dfrac{dx}{x\sqrt{a^2+x^2}}$; whence . . . . . .

$n = \dfrac{1}{2a}\,l\,\dfrac{\sqrt{a^2+x^2}-a}{x} - \dfrac{\sqrt{a^2+x^2}}{2x^2}$.

Ex. 2. $\int(a+bx^n)^p x^{m-1}dx = \text{F}$ in terms of . . . . . .
$\int(a+bx^n)^{p-1}x^{m+n-1}dx = \text{B}$.

Let $d\text{A} = (a+bx^n)^{p-1}x^{m-1}dx$; then $\text{F} = a\text{A} + b\text{B}$; and to find A in terms of B, assume $\text{P} = (a+bx^n)^p x^m$; then $d\text{P} = npbd\text{B} + m(a+bx^n)^p x^{m-1}dx = (np+m)bd\text{B} + mad\text{A}$,

$\therefore a\text{A} = \dfrac{\text{P}}{m} - \dfrac{np+m}{m}b\text{B}$; $\therefore \text{F} = \dfrac{\text{P}}{m} - \dfrac{npb\text{B}}{m}$, or . . . .

$\int(a + bx^n)^p x^{m-1}dx \cdot \cdot \cdot$

$= \dfrac{x^m(a+bx^n)^p}{m} - \dfrac{npb}{m}\int(a + bx^n)^{p-1}x^{m+n-1}dx.$

Similarly it may be shown that $\int(a + bx^n)^p x^{m-1}dx \cdot \cdot \cdot$

$= \dfrac{x^{m-n}(a+bx^n)^{p+1}}{nb(p+1)} - \dfrac{m-n}{nb(p+1)}\int(a + bx^n)^{p+1}x^{m-n-1}dx.$

Ex. 3. Required $\int \dfrac{dx}{(1+x^2)^3}$ between $x = 0$, $x = 1$.

Let $d\text{A} = \dfrac{dx}{1+x^2}$, $d\text{B} = \dfrac{dx}{(1+x^2)^2}$, $d\text{C} = \dfrac{dx}{(1+x^2)^3}$; then $\cdot \cdot \cdot$

$\text{A} = \text{B} + \int \dfrac{x^2dx}{(1+x^2)^2}$. Assume $\text{P} = x(1 + x^2)^{-1}$; then $\cdot \cdot \cdot$

$d\text{P} = d\text{A} - \dfrac{2x^2dx}{(1+x^2)^2}$, $\therefore \int \dfrac{x^2dx}{(1+x^2)^2} = \frac{1}{2}(\text{A} - \text{P})$, or $\cdot \cdot \cdot$

$\text{A} = \text{B} + \frac{1}{2}(\text{A} - \text{P})$, $\therefore \text{B} = \frac{1}{2}(\text{A} + \text{P})$.

Also $\text{B} = \text{C} + \int \dfrac{x^2dx}{(1+x^2)^3}$. Assume $\text{Q} = x(1+x^2)^{-2}$; then

$d\text{Q} = d\text{B} - \dfrac{4x^2dx}{(1-x^2)^3}$, or $\int \dfrac{x^2dx}{(1-x^2)^3} = \dfrac{1}{4}(\text{B} - \text{Q})$, $\cdot \cdot \cdot$

$\therefore \text{C} = \dfrac{3\text{B}}{4} + \dfrac{\text{Q}}{4} = \dfrac{3}{8}(\text{A} + \text{P}) + \dfrac{\text{Q}}{4}$; whence $\cdot \cdot \cdot \cdot$

$(u) = \dfrac{3}{8}(\text{A}) + \dfrac{3}{2.8} + \dfrac{1}{4.4} = \dfrac{3}{8} \cdot \dfrac{\pi}{4} + \dfrac{1}{4}.$

Ex. 4. Required the fluent of $(1 - x^2)^{\frac{2n-1}{2}}dx$ between $x = 0$, $x = 1$.

Let $d\text{A} = (1-x^2)^{\frac{1}{2}}dx$; $d\text{B} = (1-x^2)^{\frac{3}{2}}dx$; $d\text{C} = (1-x^2)^{\frac{5}{2}}dx$;

$\ldots d\text{K} = (1 - x^2)^{\frac{2n-3}{2}}dx$; $d\text{L} = (1 - x^2)^{\frac{2n-1}{2}}dx$; then $\cdot \cdot \cdot$

$\text{B} = \text{A} - \int(1 - x^2)^{\frac{1}{2}}x^2dx$. Assume $\text{P} = x(1 - x^2)^{\frac{3}{2}}$; then

$d\text{P} = d\text{B} - 3(1 - x^2)^{\frac{1}{2}}x^2dx$; $\therefore \int(1 - x^2)^{\frac{1}{2}}x^2dx \cdot \cdot \cdot \cdot$

$= \dfrac{\text{B}-\text{P}}{3}$, or $\text{A} = \dfrac{4\text{B}}{3} - \dfrac{\text{P}}{3}$.

$\text{C} = \text{B} - \int(1 - x^2)^{\frac{3}{2}}x^2dx$. Assume $\text{Q} = x(1 - x^2)^{\frac{5}{2}}$; $\cdots$

then $d\text{Q} = d\text{C} - 5(1 - x^2)^{\frac{3}{2}}x^2dx$, $\therefore \int(1 - x^2)^{\frac{3}{2}}x^2dx \cdot \cdot \cdot$

$= \frac{\text{C} - \text{Q}}{5}$, or $\text{B} = \frac{6\text{C}}{5} - \frac{\text{Q}}{5}$.

. . . . . .

$\text{L} = \text{K} - \int (1 - x^2)^{\frac{2n-3}{2}} x^2 dx$. Assume $\text{T} = x(1 - x^2)^{\frac{2n-1}{2}}$;

then $d\text{T} = d\text{L} - (2n - 1)(1 - x^2)^{\frac{2n-3}{2}} x^2 dx$; . . . .

$\therefore \int (1 - x^2)^{\frac{2n-3}{2}} x^2 dx = \frac{\text{L} - \text{T}}{2n-1}$, or $\text{K} = \frac{2n\text{L}}{2n-1} - \frac{\text{T}}{2n-1}$.

But P, Q . . . T each $= 0$ both when $x = 0$ and when $x = 1$; wherefore $(\text{A}) = \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdots \frac{2n}{2n-1} (\text{L})$. Also . . .

$(\text{A}) = \frac{\pi}{4}$; $\therefore (\text{L}) = \frac{3.5 \cdots 2n-1}{4.6 \cdots 2n} \cdot \frac{\pi}{4}$.

The whole fluent between $x = -1$, $x = +1$ is . . . .

$\frac{3.5 \cdots\cdots 2n-1}{4.6 \cdots\cdots 2n} \cdot \frac{\pi}{2}$.

Ex. 5. The form $(a + bx^n)^p (g + hx^n)^q x^{2n-1} dx = du$ is reducible by substituting $y = g + hx^n$ to . . . . . $(a' + b'y)^p (y - g)^{r-1} y^q dy$; and is therefore integrable if p and r be positive integers. If p and r be the halves of odd numbers, du may be reduced by means of the preceding articles to logarithmick or circular forms.

Similarly, if q and r be positive integers, or if they be the halves of odd numbers, the proposed form is integrable.

30. *A trinomial of the form* $\text{X}^p x^v dx$ *where* $\text{X} = a + bx^n + cx^{2n}$ *may be reduced to two of the form* $\text{X}^p x^{v+n} dx$, $\text{X}^p x^{v+2n} dx$.

For let $d\text{A} = \text{X}^p x^v dx$, $d\text{B} = \text{X}^p x^{v+n} dx$ and $d\text{C} = \text{X}^p x^{v+2n} dx$.

Assume $\text{P} = \text{X}^{p+1} x^{v+1}$; then . . .

$d\text{P} = (p+1)\{nl\, d\text{B} + 2nc\, d\text{C}\} + (v+1)\{a\, d\text{A} + b\, d\text{B} + c\, d\text{C}\}$;

and integrating,

$\text{P} = (v+1)a\text{A} + ((p+1)n + v + 1)b\text{B} + ((p+1)2n + v + 1)c\text{C}$.

Let $v + 1 = m$; then we have . . . . . .

$\int \text{X}^p x^{m-1} dx = \frac{\text{X}^{p+1} x^m}{ma} - \frac{(np + n + m)b}{ma} \int \text{X}^p x^{m+n-1} dx - \ldots$

$\frac{(2np + 2n + m)c}{ma} \int \text{X}^p x^{m+2n-1} dx$.

In order to obtain $\int x^p x^{m-1} dx$ in terms of $\int x^p x^{m-n-1} dx$, and of $\int x^p x^{m-2n-1} dx$; let $v = m - 2n - 1$, or $v + 1 = m - 2n$; then we have

P, which $= x^{p+1} x^{m-2n}$, $= (m - 2n)a \int x^p x^{m-2n-1} dx + \ldots$ $(np - n + m)b \int x^p x^{m-n-1} dx + (2np + m)c \int x^p x^{m-1} dx$, or

$$\int x^p x^{m-1} dx = \frac{x^{p+1} x^{m-2n}}{(2np+m)c} - \frac{(np-n+m)b}{(2np+m)c} \int x^p x^{m-n-1} dx - \ldots$$

$$\frac{(m-2n)a}{(2np+m)c} \int x^p x^{m-2n-1} dx.$$

Cor. By continued reduction the form $\int x^p x^{m-1} dx$ may be reduced to two of the form $\int x^p x^{m+rn-1} dx$ and $\int x^p x^{m+(r+1)n-1} dx$, where r is an integer either positive or negative.

The two reductions in this article are Hirsch's Table 2, Formulæ 4 and 2. If the index of the trinomial is $p + r$, where r is an integer, it may be reduced to p as in the case of a binomial form; but the index of x without the radical in the resulting forms will be increased by n or $2n$ or some multiple of n; and by elimination these may be expressed in terms of any two forms that may be proposed. As examples, we shall take the three remaining formulæ of Table 2.

31. *Required to find* $\int x^p x^{m-1} dx$ *in terms of* $\int x^{p-1} x^{m+n-1} dx$, *and of* $\int x^{p-1} x^{m+2n-1} dx$.

Let $dA = x^{p-1} x^{m-1} dx$; $dB = x^{p-1} x^{m+n-1} dx$; . . . . $dC = x^{p-1} x^{m+2n-1} dx$.

Assume $P = x^p x^m$; then $dP = mad A + (np + m)bdB + (2np + m)cdC$; but $x^p x^{m-1} dx = adA + bdB + cdC$; wherefore, eliminating adA, $dP = mx^p x^{m-1} dx + npbdB + 2npcdC$

or $\int x^p x^{m-1} dx =$ . . . . . .

$$\frac{x^p x^m}{m} - \frac{npb}{m} \int x^{p-1} x^{m+n-1} dx - \frac{2npc}{m} \int x^{p-1} x^{m+2n-1} dx.$$

32. *Required to find* $\int x^p x^{m-1} dx$ *in terms of* $\int x^{p-1} x^{m-1} dx$ *and of* $\int x^{p-1} x^{m+n-1} dx$.

Substituting as before, and eliminating cdC, there results

$\int x^p x^{m-1} dx =$ . . . . . .

$$\frac{x^p x^m}{2np+m} + \frac{2npa}{2np+m} \int x^{p-1} x^{m-1} dx + \frac{npb}{2np+m} \int x^{p-1} x^{m+n-1} dx.$$

33. *Required to find* $\int x^p x^{m-1} dx$ *in terms of* $\int x^{p+1} x^{m-1} dx$ *and of* $\int x^{p+1} x^{m+n-1} dx$.

Let $dA = x^p x^{m-1} dx$; $dB = x^p x^{m+n-1} dx$; $dC = x^p x^{m+2n-1} dx$; $dE = x^p x^{m+3n-1} dx$.

Then $\int x^{p+1} x^{m-1} dx = aA + bB + cC$ (1) and $\int x^{p+1} x^{m+n-1} dx = aB + bC + cE$ (2).

Also

$x^{p+1} x^m = maA + ((p+1)n + m)bB + ((p+1)2n + m)cC$ (3) and $x^{p+1} x^{m+n} = (m+n)aB + ((p+2)n + m)bC + \cdots ((2p+3)n + m)cE$ (4); from which four equations B, C and E are to be eliminated.

From (2) and (4) eliminate cE and designate the result by (5).

From (1) and (3) eliminate cC; and from (1) and (5) eliminate C; equate the resulting values of $(p+1)pB$, and we shall have

$$\int x^p x^{m-1} dx = \frac{Ax^m + Bx^{m+n}}{K} x^{p+1} + \frac{1}{K} \int (Cx^{m-1} + Dx^{m+n-1}) x^{p+1} dx,$$

where $A = 2ac - b^2$; $B = -bc$; $C = n(p+1)(b^2 - 4ac) - m(2ac - b)$; $D = ((2p+3)n + m)bc$; . . . . . $K = (p+1)(b^2 - 4ac)na$.

If the *form* of the expression for $\int x^p x^{m-1} dx$ had been known, the coefficients A, B, ... might have been found by differentiation and equating the coefficients of like terms.

Trinomials may be also integrated by reducing them to binomial forms by taking away the second term; of which examples have been given, vol. i. ch. 2.

34. *If in a trinomial of the form* $\int x^p x^{m-1} dx$, *m is a positive multiple of n, the integration can be effected.*

For substitute $y = x^n + \frac{b}{2c}$; then $x = cy^2 + a - \frac{b^2}{4c} =$ by substitution $c(y^2 + k^2)$. Also $x^n = y - \frac{b}{2c}$; whence . . . $x^{m-1} dx = \frac{1}{n} \left(y - \frac{b}{2c} \right)^{\frac{m}{n} - 1} dy$, and we have . . . . . $u = \frac{c^p}{n} \int (y^2 + k^2)^p \left(y - \frac{b}{2c} \right)^{\frac{m}{n} - 1} dy$; which, if $\frac{m}{n}$ is a positive integer, can be resolved into binomials of the form $\int (y^2 + k^2)^p y^r dy$, and is therefore integrable.

Cor. Since du may be put under the form . . . . . $(ax^{-2n} + bx^{-n} + c)^p x^{m+2pn-1}dx$, substitute $y = x^{-n} + \frac{b}{a}$ and $\mathrm{x} = ay^2 + c - \frac{b^2}{4a} = a(y^2 + k^2)$. Also $x = \left(y - \frac{b}{a}\right)^{-\frac{1}{n}}$; whence $u = -\frac{a^p}{n}\int (y^2 + k^2)^p \left(y - \frac{b}{2a}\right)^{-\frac{m+2pn+n}{n}} dy$; . . . or the proposed form is likewise integrable when $\frac{m+2pn+n}{n}$ is a negative integer.

Ex. $\int \frac{dx}{x^2\sqrt{a+bx+cx^2}} = \frac{b}{2a^2}\int \frac{dy}{y^2+k^2} - \frac{1}{a}\int \frac{ydy}{y^2+k^2}$. . . where $y = x^{-1} + \frac{b}{2a}$, and $k^2 = \frac{4ac-b^2}{a^2}$.

35. *A quadrinomial of the form* $\int x^p x^v dx$ *where* . . . $\mathrm{x} = a + bx^n + cx^{2n} + ex^{3n}$ *may be reduced to three of the form* $\int \mathrm{x}^p x^{v+n}dx$, $\int \mathrm{x}^p x^{v+2n}dx$, $\int \mathrm{x}^p x^{v+3n}dx$.

Let $d\mathrm{A} = \mathrm{x}^p x^v dx$; $d\mathrm{B} = \mathrm{x}^p x^{v+n}dx$; $d\mathrm{C} = \mathrm{x}^p x^{v+2n}dx$; . . . $d\mathrm{E} = \mathrm{x}^p x^{v+3n}dx$.

Assume $\mathrm{P} = \mathrm{x}^{p+1}x^{v+1}$; then we have

$d\mathrm{P} = (p+1)\{nbd\mathrm{B} + 2ncd\mathrm{C} + 3ned\mathrm{E}\} +$ . . . . . $(v+1)\{ad\mathrm{A} + bd\mathrm{B} + cd\mathrm{C} + ed\mathrm{E}\}$ or $\mathrm{P} = ma\mathrm{A} +$ . . . $(np+n+m)b\mathrm{B} + (2np+2n+m)c\mathrm{C} + (3np+3n+m)e\mathrm{E}$, if $v = m - 1$.

By means of this equation, and by elimination, all the formulæ of Table 3 may be deduced. The fourth Table contains the corresponding formulæ for multinomials. See also T. Simpson's Fluxions, vol. ii. pp. 138.

The difficulty of integration increases with the number of the terms within the radical. We shall conclude this part of the subject with a trinomial form which may be always rendered rational.

36. *The form in which* $\frac{du}{dx}$ *is a rational function of* x *and* $\sqrt{a+bx+cx^2}$ *may be rationalized.*

For first, let the roots of $a + bx + cx^2 = 0$ be real; and let $a + bx + cx^2 = (p - qx)(p' - q'x)$.

Assume $a + bx + cx^2 = (p-qx)^2y^2$; then $(p-qx)^2y^2 = (p - qx)(p' - q'x)$ and $y^2 = \frac{p'-q'x}{p-qx}$, or $x = \frac{py^2-p'}{qy^2-q'}$, $\therefore \cdots$ $dx = \frac{2(p'q-pq')ydy}{(qy^2-q')^2}$.

Also $\sqrt{a + bx + cx^2} = (p - qx)y = \frac{p'q-pq'}{qy^2-q}y$; $\cdots$ whence, by substitution, du becomes rationalized.

Next let the roots of $a + bx + cx^2 = 0$ be imaginary.

Here assume $a+bx+cx^2=(c^{\frac{1}{2}}x+y)^2$ or $x=\frac{y^2-a}{b-2c^{\frac{1}{2}}y}$, $\therefore$ $dx = \frac{2(by-c^{\frac{1}{2}}y^2 - ac^{\frac{1}{2}})dy}{(b - 2c^{\frac{1}{2}}y)^2}$. Also $\sqrt{a+bx+cx^2}=c^{\frac{1}{2}}x+y$ $= \frac{by - c^{\frac{1}{2}}y^2 - ac^{\frac{1}{2}}}{b-2c^{\frac{1}{2}}y}$; which substitutions will rationalize du.

This latter substitution may be used even when the roots are real, provided that c is positive.

For other forms still more general, the reader must consult the works of Euler, Lagrange, and Legendre; the result of whose labours Lacroix has collected in his Calc. Int. tom. ii. pp. 48.

The integration of du, when $\frac{du}{dx}$ is a rational function of x and $\sqrt{a + bx + cx^2 + gx^3 + hx^4}$ is made to depend upon that of the three forms $\frac{dx}{\text{R}}$, $\frac{x^2dx}{\text{R}}$, and $\frac{dx}{(x^2+a)\text{R}}$ where $\text{R} = \sqrt{\alpha + \beta x^2 + \gamma x^4}$.

37. *Transcendentals.*

These are to be integrated either by the method of Indeterminate Coefficients or by that of Continuation. The same fluent may be frequently developed in two or more different series; but that is to be taken which, under the circumstances of the case, will either terminate or converge the most rapidly.

38. *Required the fluent of* $x^n a^x dx = du$.

Assume $u = a^x(\text{A}x^n + \text{B}x^{n-1} + \text{C}x^{n-2} + \cdots)$; then

$$\frac{du}{dx}, \text{ which } = a^x x^n, = la.a^x(\text{A}x^n + \text{B}x^{n-1} + \text{C}x^{n-2}\ldots) + a^x(n\text{A}x^{n-1} + (n-1)\text{B}x^{n-2}\ldots) \Big\}; \quad \cdots$$

wherefore (Alg. 347), $\text{A}la = 1$; $\text{B}la + n\text{A} = 0$; $\cdots$

$$\text{C}la + (n-1)\text{B} = 0; \ \ldots \text{ or } \text{A} = \frac{1}{la}; \ \text{B} = -\frac{n}{la^2}; \quad \cdots$$

$$\text{C} = \frac{n(n-1)}{la^3}; \ \cdots \text{or } u = \frac{a^x}{la}\left\{x^n - \frac{nx^{n-1}}{la} + \frac{n(n-1)x^{n-2}}{la^2} - \cdots\right\};$$

a series which will terminate when n is a positive integer.

In this example it appears from its form that the coefficients are constant quantities; but in general it is necessary to consider them to be variable.

To integrate this form by the method of continuation, we have

$$\int x^n.a^x dx = x^n.\frac{a^x}{la} - \frac{n}{la}\int x^{n-1}.a^x dx;$$ and using this as a formula, we have

$$\int x^{n-1}.a^x dx = x^{n-1}.\frac{a^x}{la} - \frac{n-1}{la}\int x^{n-2}.a^x dx,$$

$$\int x^{n-2}.a^x dx = x^{n-2}.\frac{a^x}{la} - \frac{n-2}{la}\int x^{n-3}.a^x dx,$$

$$\ldots = \ldots\ldots\ldots\ldots$$

$$\text{whence } u = \frac{a^x}{la}\left\{x^n - \frac{nx^{n-1}}{la} + \frac{n(n-1)x^{n-2}}{la^2} - \ldots\right\}$$

Cor. $\int x^n e^{mx} dx \ldots$

$$= \frac{e^{mx}}{m}\left\{x^n - \frac{nx^{n-1}}{m} + \frac{n(n-1)x^{n-2}}{m^2} - \text{\&c.}\right\}.$$

39. *Required the fluent of* $x^{-n}a^x dx = du$.

If we suppose n to be negative in the former series, it will not terminate; but since $du = a^x.x^{-n}dx$, we have

$$\int a^x.x^{-n}dx = -\frac{a^x}{(n-1)x^{n-1}} + \frac{la}{n-1}\int a^x.x^{-n+1}dx; \quad \cdots$$

from which

$$\int a^x . x^{-n+1} dx = -\frac{a^x}{(n-2)x^{n-2}} + \frac{la}{n-2}\int a^x . x^{-n+2} dx,$$

$$\int a^x . x^{-n+2} dx = -\frac{a^x}{(n-3)x^{n-3}} + \frac{la}{n-3}\int a^x . x^{-n+3} dx;$$

whence $u = -\frac{a^x}{(n-1)x^{n-1}} - \frac{la.a^x}{(n-1)(n-2)x^{n-2}} - \ldots$

$$\frac{la^2.a^x}{(n-1)(n-2)(n-3)x^{n-3}} + \frac{la^3}{(n-1)(n-2)(n-3)}\int a^x x^{-n+3} dx.$$

This series cannot be continued beyond n terms, for the $n+1$th term fails. The nth term is $\frac{la^{n-1}}{(n-1)(n-2)\ldots 2.1}\int \frac{a^x dx}{x}$.

There is not any known method of integrating $\frac{a^x dx}{x}$, except by expanding a^x by the exponential theorem, and integrating each term separately.

If n is a fraction, either positive or negative, both this and the preceding series are interminable; but either furnishes an analytical value of u.

The form $\int a^x \mathrm{P} dx$, where P is any algebraick function of x, may be integrated by the method of continuation in two series by considering it either as $\int \mathrm{P} . a^x dx$ or as $\int a^x . \mathrm{P} dx$.

If P is an algebraick function of a^x, the form may be changed into one which is not transcendental by substituting $a^x = z$.

40. *Required the fluent of* $lx^m . x^{n-1} dx = du$.

Let $z = lx$; and assume $u = \mathrm{A}z^m + \mathrm{B}z^{m-1} + \mathrm{C}z^{m-2} \ldots$ where A, B, C $\ldots$ are variable; then du, which

$$= z^m x^{n-1} dx, = z^m d\mathrm{A} + z^{m-1} d\mathrm{B} + z^{m-2} d\mathrm{C} \ldots$$
$$+ m\mathrm{A}z^{m-1}\frac{dx}{x} + (m-1)\mathrm{B}z^{m-2}\frac{dx}{x} \ldots$$

or $(d\mathrm{A} - x^{n-1}dx)z^m + \left(d\mathrm{B} + \frac{m\mathrm{A}dx}{x}\right)z^{m-1} + \ldots$

$\left(d\mathrm{C} + \frac{(m-1)\mathrm{B}dx}{x}\right)z^{m-2} \ldots = 0$; wherefore (Alg. 347) $\cdots$

$d\mathrm{A} - x^{n-1}dx = 0$; $d\mathrm{B} + \frac{m\mathrm{A}dx}{x} = 0$; $d\mathrm{C} + \frac{(m-1)\mathrm{B}dx}{x} = 0$; $\ldots$

or $\mathrm{A} = \frac{x^n}{n}$; $\mathrm{B} = -\frac{mx^n}{n^2}$; $\mathrm{C} = \frac{m(m-1)x^n}{n^3}$; $\ldots$ whence

$$u = \frac{x^n}{n}\left\{ z^m - \frac{mz^{m-1}}{n} + \frac{m(m-1)z^{m-2}}{n^2} - \ldots \right.$$
$$\left. \frac{m(m-1)(m-2)z^{m-3}}{n^3} + \ldots \right\}.$$

This series will terminate if m be a positive integer, whatever be the nature of n, the value $n = 0$ excepted; in which case we have $du = lx^m dlx$; wherefore $u = \frac{lx^{m+1}}{m+1}$.

Cor. When m is a positive integer, the last term of the series is $\pm \frac{1.2.3...m}{n^{m+1}} x^n$; and consequently $\int lx^m . x^{m-1} dx$, if we suppose $x = 1$, in which case $lx = 0$, becomes . . . $\pm \frac{1.2.3...m}{n^{m+1}}$ according as m is even or odd.

Since $lx = -l\frac{1}{x}$, we have $lx^m = \pm\left(l\frac{1}{x}\right)^m$ according as m is even or odd; and consequently $\left(\int\left(l\frac{1}{x}\right)^m . x^{n-1}dx\right) = + \frac{1.2.3...m}{n^{m+1}}$ in either case.

To integrate the proposed form by the method of continuation, we have

$\int z^m . x^{n-1}dx = \frac{z^m x^n}{n} - \frac{m}{n}\int z^{m-1}x^{n-1}dx$; from which

$\int z^{m-1}.x^{n-1}dx = \frac{z^{m-1}x^n}{n} - \frac{m-1}{n}\int z^{m-2}x^{n-1}dx,$

$\int z^{m-2}.x^{n-1}dx = \frac{z^{m-2}x^n}{n} - \frac{m-2}{n}\int z^{m-3}x^{n-1}dx,$

. . . . = ...

whence $u = \frac{x^n}{n}\left\{ z^m - \frac{mz^{m-1}}{n} + \frac{m(m-1)}{n^2}z^{m-2} - \ldots . \right.$

$\left. \pm \frac{m(m-1)...2.1}{n^m} \right\}.$

Ex. 1. Required the fluent of $v^m x^n dx = du$, where $v = l . \frac{1}{x^2}$.

Here $v = -l.x^2 = -2lx$; $\therefore du = (-2)^m l x^m x^n dx$; $\therefore$ by the article

$$u = (-2)^m \cdot \frac{x^{n+1}}{n+1}\left\{ z^m - \frac{mz^{m-1}}{n+1} + \frac{m(m-1)z^{m-2}}{(n+1)^2} \cdots\cdots \pm \frac{m(m-1)\ldots 2.1}{(n+1)^m}\right\}.$$

Ex. 2. $\int x^r dx$ between $x = 0$, $x = 1$. . . .

$$du = dx\left\{ 1 + lx.\frac{x}{1} + lx^2.\frac{x^2}{1.2} + lx^3.\frac{x^3}{1.2.3} + \cdots\right\}.$$

Also by the article, we have

$$\int lx \,.\, xdx = \frac{x^2}{2}\left\{ lx - \frac{1}{2}\right\}.$$

$$\int lx^2 \,.\, x^2 dx = \frac{x^3}{3}\left\{ lx^2 - \frac{2}{3}lx + \frac{2.1}{3^2}\right\}.$$

$$\int lx^3 \,.\, x^3 dx = \frac{x^4}{4}\left\{ lx^3 - \frac{3}{4}lx^2 + \frac{3.2}{4^2}lx - \frac{3.2.1}{4^3}\right\}.$$

. . . = ...

whence $u = x - \frac{x^2}{2^2} + \frac{x^3}{3^3} - \frac{x^4}{4^4} + \cdots$

$$+ lx\left\{ \frac{x^2}{2} - \frac{x^3}{3^2} + \frac{x^4}{4^3} - \cdots\right\}$$

$$+ lx^2\left\{ \frac{x^3}{1.2.3} - \frac{x^4}{1.2.4^2} + \cdots\right\}.$$

. . . . . .

But (vol. i. ch. 5, 15, ex. 10, cor. 1) $x^n \,.\, lx^r = 0$ when $x = 0$; and $lx = 0$ when $x = 1$; wherefore

$$(u) = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

Similarly it may be shown that

$$\left(\int x^{nx} x^{m-1} dx\right) = \frac{1}{m} - \frac{n}{(m+1)^2} + \frac{n^2}{(m+2)^3} - \frac{n^3}{(m+3)^4} + \text{\&c.}$$

41. *Required the fluent of* $lx^{-m} \,.\, x^{n-1}dx = du$.

Here $du = z^{-m}x^{n-1}dx = x^n \,.\, z^{-m}dz$; wherefore we have

$$\int x^n . z^{-m} dz = -\frac{x^n}{(m-1)z^{m-1}} + \frac{n}{m-1}\int \frac{x^n . dz}{z^{m-1}}, \text{ by which}$$

$$\int \frac{x^n dz}{z^{m-1}} = -\frac{x^n}{(m-2)z^{m-2}} + \frac{n}{m-2}\int \frac{x^n dz}{z^{m-2}},$$

$$\int \frac{x^n dz}{z^{m-2}} = -\frac{x^n}{(m-3)z^{m-3}} + \frac{n}{(m-3)} \int \frac{x^n dz}{z^{m-3}};$$

whence $u = -\frac{x^n}{(m-1)z^{m-1}} - \frac{nx^n}{(m-1)(m-2)} - \ldots$

$$\frac{n^2 x^n}{(m-1)(m-2)(m-3)} + \frac{n^3}{(m-1)(m-2)(m-3)} \int \frac{x^n dz}{z^{m-3}}.$$

If this series be continued to m terms, the mth term is

$$\frac{n^{m-1}}{(m-1)(m-2)\ldots 2.1} \int \frac{x^n dz}{z}.$$

To integrate $\frac{x^n dz}{z}$, substitute $x^n = e^v$; then $v = nz$ and

$$\frac{dz}{z} = \frac{dv}{v}; \text{ or } \frac{x^n dz}{z} = \frac{e^v dv}{v} = \frac{dv}{v}\left\{1 + v + \frac{1}{1.2}v^2 + \frac{1}{1.2.3}v^3 + \ldots\right\}$$

whence $\int \frac{x^n dz}{z} = lv + v + \frac{v^2}{1.2^2} + \frac{v^3}{1.2.3^2} \ldots$

$$= l^2 . x^n + nlx + \frac{n^2}{1.2^2} lx^2 + \frac{n^3}{1.2.3^2} lx^3 \ldots$$

Cor. If $n = 0$, the series fails; but we have

$du = \frac{dx}{xlx^m} = \frac{dlx}{lx^m}$; whence $u = -\frac{1}{(m-1)lx^{m-1}}$.

If m also $= 1$, this expression for u fails; but here

$du = \frac{dx}{xlx} = \frac{dlx}{lx}$; whence $u = l^2 x$.

Similarly we may integrate $lx^m . \text{P}dx$ when P is an algebraick function of x, and m is a positive or negative integer.

By the same method of continuation we may integrate the form $l\text{Q} . \text{P}dx$, where P and Q are algebraick functions of x; for $\int l\text{Q} . \text{P}dx = l\text{Q} \int \text{P}dx - \int \frac{d\text{Q}}{\text{Q}} \int \text{P}dx$.

42. EXAMPLES.

Ex. 1. $a^x . \frac{xdx}{(b+x)^2} = du.$

Substitute $b + x = y$; then $du = \frac{a^{y-b}(y-b)dy}{y^2} \ldots$

$$= \frac{1}{a^b}\left\{\frac{a^y dy}{y} - \frac{ba^y dy}{y^2}\right\}; \text{ but } \int \frac{a^y dy}{y^2} = \int a^y . y^{-2} dy \ldots$$

$= -a^y y^{-1} + la\int a^y y^{-1} dy$; $\therefore$ . . . . . .

$$u = \frac{1}{a^b}\left\{(1 - bla)\int \frac{a^y dy}{y} + \frac{ba^y}{y}\right\}.$$

Cor. If $a = e$ and $b = 1$, we have $\int \frac{e^x . xdx}{(1+x)^2}$ . . .

$= \frac{e^x}{1+x}$ (vol. i. ch. 2, 64, ex. 5).

Ex. 2. $\frac{-dx}{z^{\frac{1}{2}}}$, where $z = l\frac{a}{x}$.

$e^z = \frac{a}{x}$ $\therefore x = ae^{-z}$ $\therefore -dx = ae^{-z}dz$ $\therefore$ . . . . .

$$du = \frac{ae^{-z}dx}{z^{\frac{1}{2}}} = \frac{adz}{z^{\frac{1}{2}}}\left\{1 - z + \frac{z^2}{1.2} - \frac{z^3}{1.2.3} + \cdots\right\}$$

$$\therefore u = a\left\{2z^{\frac{1}{2}} - \frac{2z^{\frac{3}{2}}}{1\times 3} + \frac{2z^{\frac{5}{2}}}{1.2\times 5} - \frac{2z^{\frac{7}{2}}}{1.2.3\times 7} + \cdots\right\}$$

$$= 2az^{\frac{1}{2}}\left\{1 - \frac{z}{1\times 3} + \frac{z^2}{1.2\times 5} - \frac{z^3}{1.2.3\times 7} + \cdots\right\}$$; a series which will converge if corrected between the values $x = a$, $x = b$, where b is nearly equal to a.

43. *Circular Forms.*

It is necessary to consider only such forms as involve the sine and the cosine, since all the trigonometrical lines may be reduced to these.

When the trigonometrical lines which appear in the proposed form are raised to positive integral powers, they can be expressed in series in terms of the multiple arc (Trig. p. 59); and the fluent can be obtained by integrating each term separately. It will be found, however, in general more convenient to integrate even these forms by the method of continuation.

By the method of continuation also we may reduce circular to algebraick forms, since the fluxion of the circular arc expressed in terms of any of the trigonometrical lines is algebraical:

Thus $\int x^n \sin.^{-1} x dx = \sin.^{-1} x . \frac{x^{n+1}}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}dx}{\sqrt{1-x^2}}$; and

generally $\int \mathrm{P} \sin.^{-1} x dx = \sin.^{-1} x \int \mathrm{P} dx - \int \frac{\int \mathrm{P} dx . dx}{\sqrt{1-x^2}}$.

Circular forms may be frequently integrated by the method of indeterminate coefficients; and we shall give instances of both these methods in the following articles.

44. *Required the fluent of* $\sin^{-1} x^n dx = du$.

Let $z = \sin.^{-1} x$, and $\sqrt{1-x^2} = \cos. z = y$; and since $du = z^n dx$, assume $u = \mathrm{A} z^n + \mathrm{B} z^{n-1} + \mathrm{C} z^{n-2} \ldots$; then

$$\left.\begin{array}{l} du, = z^n dx, = z^n d\mathrm{A} + z^{n-1} d\mathrm{B} + z^{n-2} d\mathrm{C} \cdots \\ \qquad + n\mathrm{A} z^{n-1} \dfrac{dx}{y} + (n-1)\mathrm{B} z^{n-2} \dfrac{dx}{y} \cdots \end{array}\right\}, \text{ or}$$

$$(d\mathrm{A} - dx) z^n + \left(d\mathrm{B} + \frac{n\mathrm{A} dx}{y}\right) z^{n-1} + \left(d\mathrm{C} + \frac{(n-1)\mathrm{B} dx}{y}\right) z^{n-2} \ldots = 0;$$

whence $d\mathrm{A} - dx = 0$; $d\mathrm{B} + \dfrac{n\mathrm{A} dx}{y} = 0$; . . . . . .

$$d\mathrm{C} + \frac{(n-1)\mathrm{B} dx}{y} = 0; \ldots \text{ or } \mathrm{A} = x; \; d\mathrm{B} = -\frac{nx dx}{\sqrt{1-x^2}} \text{ or } \ldots$$

$$\mathrm{B} = ny; \; d\mathrm{C} = -n(n-1) dx \text{ or } \mathrm{C} = -n(n-1)x; \ldots$$

$$d\mathrm{D} = n(n-1)(n-2) \frac{x dx}{\sqrt{1-x^2}} \text{ or } \mathrm{D} = -n(n-1)(n-2)y \ldots$$

wherefore

$$\left.\begin{array}{l} u = x(z^n - n(n-1) z^{n-2} + n(n-1)(n-2)(n-3) z^{n-4} \ldots) \\ \quad + y(nz^{n-1} - n(n-1)(n-2) z^{n-3} \ldots \ldots \ldots \ldots \\ \quad + n(n-1)(n-2)(n-3)(n-4) z^{n-5} \ldots) \end{array}\right\}$$

Similarly, if $z = v.s^{-1}$. w and $du = z^n dw$, it may be shown that

$$\left.\begin{array}{l} u = -y(z^n - n(n-1) z^{n-2} + n(n-1)(n-2)(n-3) z^{n-4} \ldots) \\ \quad + x(nz^{n-1} - n(n-1)(n-2) z^{n-3} \ldots \ldots \ldots \ldots \\ \quad + n(n-1)(n-2)(n-3)(n-4) z^{n-5} \ldots) \end{array}\right\}$$

where as before $x = \sin. z$ and $y = \cos. z$.

These forms may be also integrated by the method of continuation; which, in general, is to be preferred, because it does not assume that the form of the series is known. Thus, to take another example, let $du = z^2 x dx$ where $z = \tan.^{-1} x$.

$$\text{Here } u = \frac{z^2 x^2}{2} - \int \frac{z x^2 dx}{1+x^2} = \frac{z^2 x^2}{2} - \int z dx \left\{1 - \frac{1}{1+x^2}\right\}$$

$$=\frac{z^2x^2}{2}-zx+\int\frac{xdx}{1+x^2}+\int zdz=\frac{z^2x^2}{2}-zx+l\sqrt{1+x^2}+\frac{z^2}{2}$$

$$=\frac{z^2s^2}{2}-zx+ls \text{ where } s=\text{sec.}.z.$$

45. *Required the fluent of* $\sin.mx\sin.nxdx=du$.

$$\left.\begin{array}{l}\cos.(A-B)=\cos.A\cos.B+\sin.A\sin.B\\ \cos.(A+B)=\cos.A\cos.B-\sin.A\sin.B\end{array}\right\}\therefore\ .\ .\ .\ .\ .\ .\ .\ .$$

$\sin.A\sin.B=\frac{1}{2}\cos.(A-B)-\frac{1}{2}\cos.(A+B)$ or . . . .

$\sin.mx\sin.nx=\frac{1}{2}\cos.(m-n)x-\frac{1}{2}\cos.(m+n)x$; whence

$du=\frac{1}{2}\cos.(m-n)xdx-\frac{1}{2}\cos.(m+n)xdx$; and, integrating,

$$u=\tfrac{1}{2}\left\{\frac{\sin.(m-n)x}{m-n}-\frac{\sin.(m+n)x}{m+n}\right\}\quad(1.)$$

Similarly it may be shown that

$$\int\cos.mx\cos.nxdx=\tfrac{1}{2}\left\{\frac{\sin.(m-n)x}{m-n}+\frac{\sin.(m+n)x}{m+n}\right\}\quad(2.)$$

$$\int\sin.mx\cos.nxdx=-\tfrac{1}{2}\left\{\frac{\cos.(m+n)x}{m+n}+\frac{\cos.(m-n)x}{m-n}\right\}\quad(3.)$$

46. *Required the fluent of* $\sin.mx\sin.nx\cos.pxdx=du$.

$$du=\frac{\cos.pxdx}{2}\left\{\cos.(m-n)x-\cos.(m+n)x\right\}\therefore$$

$$u=\tfrac{1}{2}\int\cos.px\cos.(m-n)xdx-\tfrac{1}{2}\int\cos.px\cos.(m+n)xdx$$

$$=\tfrac{1}{4}\left\{\frac{\sin.(p-m+n)x}{p-m+n}+\frac{\sin.(p+m-n)x}{p+m-n}\right\}$$

$$-\tfrac{1}{4}\left\{\frac{\sin.(p-m-n)x}{p-m-n}+\frac{\sin.(p+m+n)x}{p+m+n}\right\}.$$

It is obvious that we can integrate by this method any form which is the product of the sines and cosines of the multiple arc of the first dimension.

47. *Required the fluent of* $\sin.^nxdx$ *and of* $\cos.^nxdx$.

$\sin.^nxdx=-\sin.^{n-1}x\,.\,d\cos.x$; whence

$$\int\sin.^nxdx=-\sin.^{n-1}x\cos.x+(n-1)\int\cos.^2x\sin.^{n-2}xdx,$$

$$=-\sin.^{n-1}x\cos.x+(n-1)\int(1-\sin.x)\sin.^{2n-2}xdx,$$

or $n\int\sin.^nxdx=-\sin.^{n-1}x\cos.x+(n-1)\int\sin.^{n-2}xdx$,

or $\int\sin.^nxdx=-\dfrac{\sin.^{n-1}x\cos.x}{n}+\dfrac{n-1}{n}\int\sin.^{n-2}xdx$, by which we have

$$\int \sin.^{n-2}x dx = -\frac{\sin.^{n-3}x \cos.x}{n-2} + \frac{n-3}{n-2}\int \sin.^{n-4}x dx,$$

$$\int \sin.^{n-4}x dx = -\frac{\sin.^{n-5}x \cos.x}{n-4} + \frac{n-5}{n-4}\int \sin.^{n-6}x dx;$$

wherefore

$$\int \sin.^{n}x dx = -\cos.x \left\{ \frac{\sin.^{n-1}x}{n} + \frac{n-1}{n(n-2)}\sin.^{n-3}x + \ldots \right.$$

$$\left. \frac{(n-1)(n-3)}{n(n-2)(n-4)}\sin.^{n-5}x \right\} + \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)}\int \sin.^{n-6}x dx.$$

In this formula substitute $\frac{\pi}{2} - x$ for x, and it becomes

$$\int \cos.^{n}x dx = \sin.x \left\{ \frac{\cos.^{n-1}x}{n} + \frac{n-1}{n(n-2)}\cos.^{n-3}x + \ldots \right.$$

$$\left. \frac{(n-1)(n-3)}{n(n-2)(n-4)}\cos.^{n-5}x \right\} + \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)}\int \cos.^{n-6}x dx.$$

These series will always terminate when n is a positive integer; for if n be odd, and $= 2m+1$, they contain $m+1$ terms, of which the $m+1$th *coefficient* will be

$$\mp \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)\ldots 3.1}.$$

If n be even, and $= 2m$, the series will contain $m+1$ terms, of which the $m+1$th will be $\frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2}x$.

Cor. When n is odd, since the fluent is expressed in terms of the sine and cosine alone, it will not exceed a certain limit when the arc is indefinitely increased; but when n is even, since the arc itself enters into the expression, the fluent is infinite when the arc is infinite.

48. *Required the fluent of* $\frac{dx}{\sin.^{n}x}$ *and of* $\frac{dx}{\cos.^{n}x}$.

$$\int \frac{dx}{\sin.^{n}x} = -\int \sin.^{-(n+1)}x d\cos.x \quad \ldots$$

$$= -\sin.^{-(n+1)}x \cos.x - (n+1)\int \sin.^{-(n+2)}x(1-\sin.^{2}x)dx,$$

$$\text{or } (n+1)\int \frac{dx}{\sin.^{n+2}x} = -\frac{\cos.x}{\sin.^{n+1}x} + n\int \frac{dx}{\sin.^{n}x},$$

$$\text{or } \int \frac{dx}{\sin.^{n+2}x} = -\frac{\cos.x}{(n+1)\sin.^{n+1}x} + \frac{n}{n+1}\int \frac{dx}{\sin.^{n}x}; \text{ in which}$$

diminish n by 2, and we have

$\int \frac{dx}{\sin.^n x} = -\frac{\cos.x}{(n-1)\sin.^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\sin.^{n-2}x}$; from which we have

$$\int \frac{dx}{\sin.^{n-2}x} = -\frac{\cos.x}{(n-3)\sin.^{n-3}x} + \frac{n-4}{n-3}\int \frac{dx}{\sin.^{n-4}x},$$

$$\int \frac{dx}{\sin.^{n-4}x} = -\frac{\cos.x}{(n-5)\sin.^{n-5}x} + \frac{n-6}{n-5}\int \frac{dx}{\sin.^{n-6}x};$$ wherefore

$$\int \frac{dx}{\sin.^n x} = -\cos.x \left\{ \frac{1}{(n-1)\sin.^{n-1}x} + \frac{n-2}{(n-1)(n-3)\sin.^{n-3}x} \right.$$
$$\left. + \frac{(n-2)(n-4)}{(n-1)(n-3)(n-5)\sin.^{n-5}x} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ \frac{(n-2)(n-4)(n-6)}{(n-1)(n-3)(n-5)}\int \frac{dx}{\sin.^{n-6}x}.$$

Similarly it may be shown that

$\int \frac{dx}{\cos.^n x} = \frac{\sin.x}{(n-1)\cos.^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos.^{n-2}x}$, from which we obtain

$$\int \frac{dx}{\cos.^n x} = \sin.x \left\{ \frac{1}{(n-1)\cos.^{n-1}x} + \frac{n-2}{(n-1)(n-3)\cos.^{n-3}x} \right.$$
$$\left. + \frac{(n-2)(n-4)}{(n-1)(n-3)(n-5)\cos.^{n-5}x} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ \frac{(n-2)(n-4)(n-6)}{(n-1)(n-3)(n-5)}\int \frac{dx}{\cos.^{n-6}x}.$$

These series will terminate when n is an integer. If n is even, the last terms will contain $\tan.x$; if odd, the last terms will contain $\int \frac{dx}{\sin.x}$ which $= l.\tan.\frac{1}{2}x$, and $\int \frac{dx}{\cos.x} \cdots$ which $= l\,\frac{\sqrt{1+\sin.^2 x}}{\sqrt{1-\sin.^2 x}}$ (vol. i. ch. 2, 64, ex. 1 and 2.)

49. *Required the fluent of* $\sin.^m x \cos.^n x dx$.

$\sin.^m x \cos.^n x dx = -\sin.^{m-1}x . \cos.^n x d\cos.x$; whence

$\int \sin.^m x \cos.^n x dx \quad . \quad . \quad . \quad .$

$$= -\frac{\sin.^{m-1}x \cos.^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin.^{m-2}x \cos.^{n+2}x dx,$$

$$= -\frac{\sin.^{m-1}x \cos.^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin.^{m-2}x \cos.^n x(1-\sin.^2 x)dx,$$

$= -\frac{\sin.^{m-1}x \cos.^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin.^{m-2}x \cos.^{n}x dx$ (A); from which we have

$\int \sin.^{m-2}x \cos.^{n}x dx \ldots$

$= -\frac{\sin.^{m-3}x \cos.^{n+1}x}{m+n-2} + \frac{m-3}{m+n-2}\int \sin.^{m-4}x \cos.^{n}x dx,$

$\int \sin.^{m-4}x \cos.^{n}x dx \ldots$

$= -\frac{\sin.^{m-5}x \cos.^{n+1}x}{m+n-4} + \frac{m-5}{m+n-4}\int \sin.^{m-6}x \cos.^{n}x dx;$

wherefore

$$\int \sin.^{m}x \cos.^{n}x dx = -\frac{\cos.^{n+1}x}{m+n}\left\{\sin.^{m-1}x + \frac{m-1}{m+n-2}\sin.^{m-3}x\right.$$

$$\left. + \frac{(m-1)(m-3)}{(m+n-2)(m+n-4)}\sin.^{m-5}x \right\} \ldots\ldots\ldots$$

$$+ \frac{(m-1)(m-3)(m-5)}{(m+n)(m+n-2)(m+n-4)}\int \sin.^{m-6}x \cos.^{n}x dx.$$

If m is a positive integer and even, the proposed form is thus reduced to $\int \cos.^{n}x dx$, which has been integrated, art. 47. If m be odd, the form is reduced to $\int \sin.x \cos.^{n}x dx$, which $= -\int \cos.^{n}x d\cos.x = -\frac{\cos.^{n+1}x}{n+1}$.

If it be more convenient to reduce the index of $\cos.x$, we have either by the same process, or by substituting in (A) $\frac{\pi}{2} - x$ for x, and interchanging m and n, ...

$\int \sin.^{m}x \cos.^{n}x dx \ldots$

$= \frac{\sin.^{m+1}x \cos.^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin.^{m}x \cos.^{n-2}x dx$ (B), from which we have

$$\int \sin.^{m}x \cos.^{n}x dx = \frac{\sin.^{m+1}x}{m+n}\left\{\cos.^{n-1}x + \frac{n-1}{m+n-2}\cos.^{n-3}x \cdots\right.$$

$$\left. + \frac{(n-1)(n-3)}{(m+n-2)(m+n-4)}\cos.^{n-5}x \right\} \ldots\ldots\ldots$$

$$+ \frac{(n-1)(n-3)(n-5)}{(m+n)(m+n-2)(m+n-4)}\int \cos.^{n-6}x \sin.^{m}x dx;$$ which reduces the proposed form, either to $\int \sin.^{m}x dx$, or to $\int \cos.x \sin.^{m}x dx = \frac{\sin.^{m+1}x}{m+1}$, according as n is even or odd.

The proposed form may be also changed into one that is algebraical by substitution.

For let $\sin.x = z$; then $\cos.x = \sqrt{1-z^2}$ and . . . $dx = \frac{dz}{\sqrt{1-z^2}}$. Whence $u = \int (1-z^2)^{\frac{n-1}{2}} z^m dz$, which may be rationalized when either $\frac{m+1}{2}$ or $\frac{m+n}{n}$ is an integer (vol. i. ch. 2. 46, and vol. ii. ch. 1. 18).

50. *Required the fluent of* $\frac{\sin.^m x dx}{\cos.^n x}$.

$$\int \frac{\sin.^m x dx}{\cos.^n x} = -\int \sin.^{m-1} x . \cos.^{-n} x d\cos.x$$

$$= \frac{\sin.^{m-1} x \cos.^{-n+1} x}{n-1} - \frac{m-1}{n-1} \int \sin.^{m-2} x \cos.^{-n} x (1 - \sin.^2 x) dx$$

$$= -\frac{\sin.^{m-1} x \cos.^{-n+1} x}{m-n} + \frac{m-1}{m-n} \int \frac{\sin.^{m-2} x}{\cos.^n x} dx \quad (C),$$

which is the same as (A), n being negative, and the proposed form is thus reducible to $\int \frac{dx}{\cos.^n x}$ or $\int \frac{\sin.x dx}{\cos.^n x} = \frac{1}{(n-1)\cos.^{n-1} x}$, according as m is even or odd.

If it be more convenient to raise the index of cos. x, we must refer to the form (B); which, when n is negative, becomes

$$\int \frac{\sin.^m x dx}{\cos.^n x} = \frac{\sin.^{m+1} x}{(m-n)\cos.^{n+1} x} - \frac{n+1}{m-n} \int \frac{\sin.^m x dx}{\cos.^{n+2} x}, \text{ or}$$

$$\int \frac{\sin.^m x dx}{\cos.^{n+2} x} = \frac{\sin.^{m+1} x}{(n+1)\cos.^{n+1} x} - \frac{m-n}{n+1} \int \frac{\sin.^m x dx}{\cos.^n x};$$

in which diminish n by 2; and we have

$$\int \frac{\sin.^m x dx}{\cos.^n x} = \frac{\sin.^{m+1} x}{(n-1)\cos.^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\sin.^m x dx}{\cos.^{n-2} x} \quad (D);$$

by which we obtain

$$\int \frac{\sin.^m x dx}{\cos.^n x} = \frac{\sin.^{m+1} x}{n-1} \left\{ \frac{1}{\cos.^{n-1} x} - \frac{m-n+2}{(n-3)\cos.^{n-3} x} \quad . \quad . \quad . \right.$$

$$\left. + \frac{(m-n+2)(m-n+4)}{(n-3)(n-5)\cos.^{n-5} x} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$- \frac{(m-n+2)(m-n+4)(m-n+6)}{(n-1)(n-3)(n-5)} \int \frac{\sin.^m x dx}{\cos.^{n-6} x};$$

which reduces the proposed form to $\int \sin.^m x dx$ if n is even; and if odd, to $\int \frac{\sin.^m x dx}{\cos.x}$.

To find $\int \frac{\sin.^m x dx}{\cos.x}$, we have

$$\int \frac{\sin.^m x dx}{\cos.x} = \int \frac{\sin.^{m-2} x dx}{\cos.x}(1 - \cos.^2 x) \quad \cdots\cdots$$

$$= \int \frac{\sin.^{m-2} x dx}{\cos.x} - \int \cos.x \sin.^{m-2} x dx \quad \cdots\cdots$$

$$= \int \frac{\sin.^{m-4} x dx}{\cos.x} - \int \cos.x \sin.^{m-4} x dx - \int \cos.x \sin.^{m-2} x dx,$$

which therefore depends upon $\int \frac{dx}{\cos.x} = l \frac{\sqrt{1 + \sin.^2 x}}{\sqrt{1 - \sin.^2 x}}$ or upon $\int \frac{\sin.x dx}{\cos.x} = l \frac{1}{\cos.x}$, according as m is even or odd.

By the same method the form $\frac{\cos.^n x dx}{\sin.^m x}$ may be integrated; or it may be reduced to the form of the article by substituting $\frac{\pi}{2} - x$ for x.

51. *Required the fluent of* $\frac{dx}{\sin.^m x \cos.^n x}$.

$$\int \frac{dx}{\sin.^m x \cos.^n x} = \int \frac{(\sin.^2 x + \cos.^2 x) dx}{\sin.^m x \cos.^n x} = \int \frac{dx}{\sin.^{m-2} x \cos.^n x} \cdots$$

$+ \int \frac{dx}{\sin.^m x \cos.^{n-2} x}$; and by repeating the process, these may be reduced to simpler forms, dependent upon the nature of m and n.

When $m = n$, $u = 2^n \int \frac{dx}{\sin.^n 2x} = 2^{n-1} \int \frac{dy}{\sin.^n y}$, if $y = 2x$; which has been found, art. 48.

Ex. 1. $du = \frac{dx}{\sin.^3 x \cos.^2 x}$.

Here $du = \frac{(\sin.^2 x + \cos.^2 x) dx}{\sin.^3 x \cos.^2 x} = \frac{dx}{\sin.x \cos.^2 x} + \frac{dx}{\sin.^3 x} \cdots$

$$= \frac{\sin.x dx}{\cos.^2} + \frac{dx}{\sin.x} + \frac{dx}{\sin.^3 x}$$

$$\therefore u = \frac{1}{\cos.x} + l \tan.\tfrac{1}{2}x + \int \frac{dx}{\sin.^3 x}.$$

But (art. 48.), $\int \frac{dx}{\sin.^3 x} = -\frac{\cos.x}{2\sin.^2 x} + \frac{1}{2}\int \frac{dx}{\sin.x}$

$$\therefore u = \frac{1}{\cos.x} + \tfrac{3}{2} l \tan.\frac{x}{2} - \frac{\cos.x}{2\sin.^2 x}.$$

Ex. 2. $du = \frac{dx}{\sin.^2 x \cos.^4 x}$.

Here $du = \frac{dx}{\cos.^4 x} + \frac{dx}{\sin.^2 x \cos.^2 x}$; but $\int \frac{dx}{\cos.^4 x}$ . . .

$$= \sin.x \left\{ \frac{1}{3\cos.^3 x} + \frac{2}{3\cos.x} \right\} = \frac{\sin.x}{3\cos.^3 x} + \tfrac{2}{3}\tan.x.$$

Also $\int \frac{dx}{\sin.^2 x \cos.^2 x} = 2\int \frac{2dx}{\sin.^2 2x} = -\frac{2\cos.2x}{\sin.2x} = -2\cot.2x.$

Ex. 3. $du = \frac{dx}{\sin.^4 x \cos.x}$.

Here $du = \frac{dx}{\sin.^2 x \cos.x} + \frac{\cos.x dx}{\sin.^4 x} = \frac{dx}{\cos.x} + \frac{\cos.x dx}{\sin.^2 x}$ . . .

$$+ \frac{\cos.x dx}{\sin.^4 x} \quad \therefore u = l \tan.\frac{\pi + 2x}{4} - \frac{1}{\sin.x} - \frac{1}{3}\frac{1}{\sin.^3 x}.$$

It appears from the preceding articles that the form $\int \sin.^m x \cos.^n x dx$ is always integrable, if m and n are integers either positive or negative. There are certain cases in which the preceding theorems fail, but they are all integrable by other methods, which are sufficiently obvious: thus, theorem (c) of art. 50 fails when $m = n$; but in this case $du = \tan.^n x dx$. Substitute $t = \tan.x$; then . . . $dx = \frac{dt}{1+t^2}$ and $du = \frac{t^n dt}{t^2+1}$, which is integrable by division, if n is positive; and if n be negative, either by the method of indeterminate coefficients, or by substituting $y = \frac{1}{t}$.

52. *Required the fluent of* $z^r y^m x^n dz = du$ *where* $y = \sin.z$, $x = \cos.z$.

To integrate by the method of indeterminate coefficients,

assume $u = Az^r + Bz^{r-1} + Cz^{r-2} + \cdots$ then

$z^r y^m x^n dz = z^r dA + z^{r-1} dB + z^{r-2} dC + \cdots$
$\quad + rAz^{r-1}dz + (r-1)Bz^{r-2}dz + \cdots$; whence

$dA = y^m x^n dz = \sin.^m z \cos.^n z dz$, which has been integrated, art. 49. Let $\int \sin.^m z \cos.^n z dz = P$; then $A = P$. Also $dB = -rPdz$; each term of which is integrable by the same article. Let $\int Pdz = Q$; then $B = -rQ$; and $dC = r(r-1)Qdz$.

Let $\int Qdz = R$; then $C = r(r-1)R$; and so on; whence we have $u = Pz^r - rQz^{r-1} + r(r-1)Rz^{r-2} - \cdots$

Cor. Since $dz = \frac{dy}{x}$, and also $= -\frac{dx}{y}$, we integrate by this method the forms $\int z^r y^m x^{n-1} dy$ and $\int z^r y^{m-1} x^n dx$.

Ex. 1. $du = \frac{zy^2 dy}{x}$.

Here $du = zy^2 dz \therefore r = 1, m = 2, n = 0$; and $\cdots$

$P = \int \sin.^2 z dz = -\frac{xy}{2} + \frac{z}{2}$ (47.)

Also $dQ = -\frac{xydz}{2} + \frac{zdz}{2} = -\frac{ydy}{2} + \frac{zdz}{2} \therefore Q = \frac{z^2 - y^2}{4}$;

and $u = -\frac{xyz}{2} + \frac{z^2}{2} - \frac{z^2 - y^2}{4} = \frac{z^2 - 2xyz + y^2}{4}$.

Ex. 2. $du = -\pi(y+z)^2 dx$; of which the fluent is the solid generated by the revolution of a cycloid round its diameter.

$du = -\pi(z^2 dx + 2zydx + y^2 dx)$.

$\int z^2 dx = -\int z^2 y dz = xz^2 - 2yz - 2x$.

$\int zydx = -\int zy^2 dz = -\frac{z^2 - 2xyz + y^2}{4}$ (Ex. 1),

$\int y^2 dx = \int (1 - x^2) dx = x - \frac{x^3}{3}$.

$\therefore u = \pi \left\{ \left(\frac{1}{2} - x\right) z^2 + (2-x)yz + \frac{y^2}{2} + x + \frac{x^3}{3} \right\} + \text{cor.}$

and $o = \pi \cdot \frac{4}{3}$.

$\therefore u = \pi \left\{ \left(\frac{1}{2} - x\right) z^2 + (2-x)yz + \frac{y^2}{2} + x + \frac{x^3}{3} - \frac{4}{3} \right\}$

and $(u) = \frac{3\pi^3}{2} - \frac{8\pi}{3}$.

53. *Required the fluent of* $\sqrt{1+y}.z^r dz$ *and of* $\cdots$ $\sqrt{1+x}.z^r dz$.

Let $du = \sqrt{1+y}.z^r dz = z^r\sqrt{1+y}\cdot\frac{dy}{\sqrt{1-y^2}}\cdots$

$= z^r\cdot\frac{dy}{\sqrt{1-y}}$; wherefore, integrating,

$u = -z^r\cdot 2\sqrt{1-y} + 2r\int z^{r-1}\sqrt{1-y}\,dz$

$= -2z^r\sqrt{1-y} + 2r\int z^{r-1}\cdot\frac{dy}{\sqrt{1+y}}$; which, when r is a positive integer, reduces u to the form . . . .

$\int\frac{dy}{\sqrt{1\pm y}} = \pm 2\sqrt{1\pm y}$.

Similarly we may integrate $\int\sqrt{1+x}.z^r dz$.

54. *Required the fluent of* $\frac{dz}{a+b\sin.z} = du$.

This may be rendered algebraical by substituting $x = \sin.z$; in which case it becomes $\frac{dx}{(a+bx)\sqrt{1-x^2}}$; which may be integrated by substituting $y = \frac{\sqrt{1-x^2}}{1+x}$, or $x = \frac{1-y^2}{1+y^2}$.

Substitute then in the proposed form, $\sin.z = \frac{1-y^2}{1+y^2}$; . . .

then $\cos.z = \frac{2y}{1+y^2}$, and differentiating, ...

$-\sin.z dz = \frac{2dy}{1+y^2} - \frac{4y^2dy}{(1+y^2)^2} = \frac{2(1-y^2)dy}{(1+y^2)^2}$, or . . . .

$dz = -\frac{2dy}{1+y^2}$, and $du = -\frac{2dy}{a+b+(a-b)y^2}$; which is a tangential or a logarithmick form according as a is greater or less than b.

By a similar substitution $\frac{dz}{a+b\cos.z}$, which is ex. 9, vol. i. ch. 2, 64, may be integrated; as also $\frac{dz}{a+b\tan.z}$; which, ...

substituting $y = \tan.z$, becomes $\frac{dy}{(a+by)(1+y^2)}$, a rational form.

55. *Required the fluents of* $\frac{\sin.z dz}{a+b\sin.z}$ *and of* $\frac{\cos.z dz}{a+b\cos.z}$.

To find $\int \frac{\sin.z dz}{a+b\sin.z}$, assume $\frac{\sin.z}{a+b\sin.z} = \text{A} + \frac{\text{B}}{a+b\sin.z}$ $= \frac{\text{A}a+\text{A}b\sin.z+\text{B}}{a+b\sin.z}$; or $\text{A}a + \text{B} + (\text{A}b - 1)\sin.z = 0$; $\cdots$ wherefore $\text{A}a + \text{B} = 0$ and $\text{A}b - 1 = 0$; whence $\text{A} = \frac{1}{b}\cdots$ and $\text{B} = -\frac{a}{b}$; and consequently the proposed form is reduced to $\int \frac{dz}{a+b\sin.z}$, which has been already integrated.

By a similar assumption, $\int \frac{\cos.x dx}{a+b\cos.x}$ may be reduced to $\int \frac{dz}{a+b\cos.z}$.

The same reductions may be also effected by actual division.

The forms $\frac{\cos.z dz}{a+b\sin.z}$ and $\frac{\sin.z dz}{a+b\cos.z}$ are the same as $\cdots$ $\frac{d\sin.z}{a+b\sin.z}$ and $-\frac{d\cos.z}{a+b\cos.z}$; and consequently their fluents are $\frac{1}{b}l(a+b\sin.z)$ and $-\frac{1}{b}l(a+b\cos.z)$.

56. *Required the fluent of* $\frac{(p+q\cos.z)dz}{(a+b\cos.z)^n}$ *and of* $\cdots$ $\frac{(p+q\sin.z)dz}{(a+b\sin.z)^n}$.

Assume $\int \frac{(p+q\cos.z)dz}{(a+b\cos.z)^n} = \frac{\text{A}\sin.z}{(a+b\cos.z)^{n-1}} \cdots$ $+ \int \frac{(\text{B}+\text{C}\cos.z)dz}{(a+b\cos.z)^{n-1}}$; then differentiating and dividing by dz, we have

$\frac{p+q\cos.z}{(a+b\cos.z)^n} = \frac{\text{A}\cos.z}{(a+b\cos.z)^{n-1}} + \frac{(n-1)b\text{A}\sin.^2z}{(a+b\cos.z)^n}$ . . .

$+ \frac{\text{B}+\text{C}\cos.z}{(a+b\cos.z)^{n-1}}$, or $p + q\cos.z = \text{A}\cos.z(a + b\cos.z)$ . .

$+ (n-1)b\text{A}(1-\cos.^2z) + (\text{B}+\text{C}\cos.z)(a+b\cos.z)$ or $o = ((2-n)\text{A}+\text{C})b\cos.^2z + (a\text{A}+b\text{B}+a\text{C}-q)\cos.z + (n-1)b\text{A} + a\text{B} - p = 0$; wherefore $(2-n)\text{A}+\text{C}=0$; $a\text{A}+b\text{B}+a\text{C}-q=0$ and $(n-1)b\text{A}+a\text{B}-p=0$.

From which we obtain $\text{C} = (n-2)\text{A}$; wherefore

$(n-1)a\text{A} + b\text{B} - q = 0$, or $\text{B} = \frac{q}{b} - (n-1)\frac{a\text{A}}{b}$;

wherefore $(n-1)b\text{A} + \frac{aq}{b} - (n-1)\frac{a^2\text{A}}{b} - p = 0$, or

$\text{A} = \frac{aq-bp}{(n-1)(a^2-b^2)}$; $\text{B} = \frac{q}{b} - \frac{a}{b}\cdot\frac{aq-bp}{a^2-b^2} = \frac{ap-bq}{a^2-b^2}$ and

$\text{C} = \frac{(n-2)(aq-bp)}{(n-1)(a^2-b^2)}$.

By this assumption, the proposed form, if n is a positive integer, is reducible to $\int\frac{(\text{B}+\text{C}\cos z)dz}{a+b\cos.z}$; which by division

$= \frac{\text{C}z}{b} + \frac{b\text{B}-a\text{C}}{b}\int\frac{dz}{a+b\cos.z}$.

By a similar assumption we may integrate $\frac{(p+q\sin.z)dz}{(a+b\sin.z)^n}$.

The forms $\int\frac{(p+q\sin.z)dz}{(a+b\cos.z)^n}$ and $\int\frac{(p+q\cos.z)dz}{(a+b\sin.z)^n}$ separate into $\int\frac{pdz}{(a+b\cos.z)^n}$, $\int\frac{q\sin.zdz}{(a+b\cos.z)^n}$ and $\int\frac{pdz}{(a+b\sin.z)^n}$, . . . $\int\frac{q\cos.zdz}{(a+b\sin.z)^n}$; of which $\int\frac{pdz}{(a+b\cos.z)^n}$ and $\int\frac{pdz}{(a+b\sin.z)^n}$, are particular cases of the article, q being $= 0$; also $\int\frac{\sin.zdz}{(a+b\cos.z)^n}$ and $\int\frac{\cos.zdz}{(a+b\sin.z)^n}$ may be changed by substitution into the forms $\pm\int\frac{dy}{(a+by)^n} = \mp\frac{1}{n-1}\cdot\frac{1}{(a+by)^{n-1}}$.

57. *Required the fluent of* $a^{mz}\sin.^n z dz$ *and of* $a^{mz}\cos.^n z dz$.

$$\int a^{mz}\sin.^n z dz \ . \ . \ .$$
$$= -\int a^{mz}\sin.^{n-1} z \cdot d\cos.z$$
$$= -a^{mz}\sin.^{n-1} z\cos.z + m\mathrm{A}\int a^{mz}\sin.^{n-1} z\cos.z dz \ . \ . \ .$$
$$+ (n-1)\int a^{mz}\sin.^{n-2} z\cos.^2 z dz$$
$$= -a^{mz}\sin.^{n-1} z\cos.z + \frac{m\mathrm{A}a^{mz}\sin.^n z}{n} - \frac{m^2\mathrm{A}^2}{n}\int a^{mz}\sin.^n z dz$$
$$+ (n-1)\int a^{mz}\sin.^{n-2} z(1-\sin.^2 z)dz,$$

$$\text{or } \frac{m^2\mathrm{A}^2+n^2}{n}\int a^{mz}\sin.^n z dz = \frac{m\mathrm{A}a^{mz}\sin.^n z}{n} - a^{mz}\sin.^{n-1} z\cos.z$$
$$+ (n-1)\int a^{mz}\sin.^{n-2} z dz,$$

$$\text{or } \int a^{mz}\sin.^n z dz = \frac{a^{mz}\sin.^{n-1} z}{m^2\mathrm{A}^2+n^2}\left\{ m\mathrm{A}\sin.z - n\cos.z \right\}$$
$$+ \frac{n(n-1)}{m^2\mathrm{A}^2+n^2}\int a^{mz}\sin.^{n-2} z dz\,;\ \text{where } \mathrm{A} = la.$$

By this formula the proposed form is reduced to $\int a^{mz} dz = \frac{a^{mz}}{\mathrm{A}}$, if n is even; and if n is odd, to $\int a^{mz}\sin.z dz$; which by the formula, n being $= 1$, is $\frac{a^{mz}}{m^2\mathrm{A}^2+1}\left\{ m\mathrm{A}\sin.z - \cos.z \right\}$.

Similarly, we may integrate $\int a^{mz}\cos.^n z dz$.

58. *Required the fluent of* $e^{az}\sin.mz\cos.nz dz = du$.

$du = \frac{e^{az}dz}{2}\left\{ \sin.(m+n)z + \sin.(m-n)z \right\}$; which reduces u to forms which have been already integrated.

59. *Required the fluent of* $e^{az}\sin.^m z\cos.^n z dz$.

In order to integrate this form, $\sin.^m z$ and $\cos.^n z$ must be developed in series in terms of the sine and cosine of the multiple arc, and each term may be then integrated separately.

These are all the forms which are usually given in elementary treatises. But in the application of the science, fluents frequently occur which are not included in any of the preceding forms, and which must be integrated by independent methods. It is also requisite that the results should be under a form convenient for computation; and we shall therefore conclude this chapter with one instance out of many which might have been selected of the necessity of

adopting certain artifices of calculation in order to obtain the fluent in a convenient form.

60. *Required the fluent of* $l(1 + n\cos.z) \,.\, dz = du$.

$l(1 + n\cos.z) = n\cos.z - \frac{1}{2}n^2\cos.^2z + \frac{1}{3}n^3\cos.^3z - \ldots$; and if each term be multiplied by dz and integrated, we shall obtain a series which $= u$; but in integrating the forms $\cos.zdz$, $\cos.^2zdz$, $\cdots$ (art. 47), the results, when added together, will not be expressed after any simple law. It is manifest that since they may be expressed in terms of the powers of $\cos.z$, they may be arranged in a series of the form $\text{A} + \text{B}\cos.z + \text{C}\cos.2z \cdots$; but this being a laborious process, Euler employs the method explained, vol. i. ch. 4, art. 11.

Let $l(1 + n\cos.z) = -\text{A} + \text{B}\cos.z - \text{C}\cos.2z + \cdots$;

then $-\dfrac{n\sin.z}{1+n\cos.z} = -\text{B}\sin.z + 2\text{C}\sin.2z - \cdots$

Clearing this of fractions, and expressing $\cos.z\sin.z$, $\cos.z\sin.2z$, $\cos.z\sin.3z \cdots$ in terms of the sines of the multiple arcs by means of the formula $\cos.z\sin.mz = \frac{1}{2}\sin.(m+1)z + \frac{1}{2}\sin.(m-1)z$, we shall have

$$-n\sin.z = -\text{B}\left|\begin{array}{l}\sin.z + 2\text{C}\\ \quad -\dfrac{n\text{B}}{2}\end{array}\right|\sin.2z - 3\text{D}\left|\begin{array}{l}\sin.2z - \cdots;\\ \quad +n\text{C}\end{array}\right.$$
$$n\text{C}\left|\begin{array}{l}\\ \quad -\dfrac{3n\text{D}}{2}\end{array}\right.\quad\left|\begin{array}{l}\\ \quad +2n\text{E}\end{array}\right.$$

wherefore $0 = -\text{B} + n\text{C} + n$ or $\text{C} = \dfrac{\text{B}-n}{n}$; $\text{D} = \dfrac{4\text{C}-n\text{B}}{3n}$;

$\text{E} = \dfrac{6\text{D}-2n\text{C}}{4n}$; and similarly, $\text{F} = \dfrac{8\text{E}-3n\text{D}}{5n}$, &c.; which expresses all the coefficients in terms of A and B.

Now to find these; we have

$l(1 + \cos.nz) = n\cos.z - \frac{1}{2}n^2\cos.^2z + \frac{1}{3}n^3\cos.^3z - \cdots$ and expressing $\cos.^2z$, $\cos.^3z \cdots$ in terms of $\cos.2z$, $\cos.3z$, $\cdots$ we shall find

$$l(1+\cos.nz) = -\left\{\frac{1}{2}\frac{n^2}{2} + \frac{1.3}{2.4}\frac{n^4}{4} + \frac{1.3.5}{2.4.6}\frac{n^6}{6} + \cdots\right\}$$
$$+2\left\{\frac{1}{2}n + \frac{1.3}{2.4}\frac{n^3}{3} + \frac{1.3.5}{2.4.6}\frac{n^5}{5} + \cdots\right\}\cos.z$$
$$- \&c.$$

or $\text{A} = \frac{1}{2}\frac{n^2}{2} + \frac{1.3}{2.4}\frac{n^4}{4} + \frac{1.3.5}{2.4.6}\frac{n^6}{6} + \cdots$

To sum this series, differentiate both sides of the equation, and multiply by n; and we have

$$\frac{n d\text{A}}{dn} = \frac{1}{2}n^2 + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \cdots$$

$$= (1 - n^2)^{-\frac{1}{2}} - 1; \text{ whence}$$

$$d\text{A} = \frac{dn}{n\sqrt{1-n^2}} - \frac{dn}{n}; \text{ and integrating,}$$

$$\text{A} = l\frac{1-\sqrt{1-n^2}}{n^2} + \text{cor.}$$

To find the correction, $\text{A} = 0$ when $n = 0$; but in this case, $l\frac{1-\sqrt{1-n^2}}{n^2}$ becomes $l\frac{0}{0}$; and to find this value, substitute $v = l\frac{1-\sqrt{1-n^2}}{n^2}$ or $e^v = \frac{1-\sqrt{1-n^2}}{n^2}$;

wherefore $(e^v) = \frac{n}{2n\sqrt{1-n^2}} = \frac{1}{2}$, or $(v) = l\frac{1}{2}$; and consequently $\text{A} = l\frac{2-2\sqrt{1-n^2}}{n^2}$.

Next to find B; we have $\frac{\text{B}}{2} = \frac{1}{2}n + \frac{1.3}{2.4}\frac{n^3}{3} + \frac{1.3.5}{2.4.6}\frac{n^5}{5} + \cdots$

whence $\frac{n^2 d\text{B}}{2dn} = \frac{1}{2}n^2 + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \ldots$

$$= (1 - n^2)^{-\frac{1}{2}} - 1,$$

or $d\text{B} = \frac{2dn}{n^2\sqrt{1-n^2}} - \frac{2dn}{n^2}$, and integrating, we have $\text{B} = -\frac{2\sqrt{1-n^2}}{n} + \frac{2}{n} = \frac{2(1-\sqrt{1-n^2})}{n}$, for correction $= 0$.

Substituting these values in the series assumed for $l(1 + \cos.nz)$, and putting for the sake of brevity . . . $m = \frac{1-\sqrt{1-n^2}}{n}$, we shall have $l(1 + \cos.nz)$ . . . .

$$= -\,l\frac{2m}{n} + 2m\cos.z - m^2\cos.2z + \frac{2m^3\cos.3z}{3}\ldots;\text{ wherefore}$$

$$u = -zl\frac{2m}{n} + \frac{2m\sin.z}{1} - \frac{2m^2\sin.2z}{4} + \frac{2m^3\sin.3z}{9} \quad . \quad . \quad .$$

$$- \frac{2m^4\sin.4z}{16} + \cdots + \text{cor.}$$

Cor. Supposing $n = +1$ and $n = -1$, in which cases $m = +1$ and $m = -1$, we have

$$l(1 + \cos.z) = -\,l2 + 2\cos.z - \cos.2z + \frac{2\cos.3z}{3} - \ldots + \text{cor.}$$

$$l(1 - \cos.z) = -\,l2 - 2\cos.z - \cos.2z - \frac{2\cos.3z}{3} - \ldots + \text{cor.}$$

But $1 + \cos.z = 2\cos.^2\frac{z}{2}$, and $1 - \cos.z = 2\sin.^2\frac{z}{2}$; and by substitution and subtraction, we obtain

$$l\tan.\frac{z}{2} = -2\left\{\cos.z + \frac{\cos.3z}{3} + \frac{\cos.5z}{5} + \frac{\cos.7z}{7} + \ldots\right\}.$$

61. MISCELLANEOUS PRAXIS.

1. $du = \dfrac{dx}{(1+x)^{\frac{1}{3}} + (1+x)^{\frac{1}{5}}}$ $\therefore u = 15\int\dfrac{z^{11}dz}{z^2+1}$, where . . . $z^{15} = 1 + x$.

2. $du = \dfrac{dx}{(a^2+x^2)^4}$. The equations for determining u are

$\text{B} = \dfrac{\text{A}+\text{P}}{2a^2}$, $\text{C} = \dfrac{3\text{B}+\text{Q}}{4a^2}$, and $u = \dfrac{5\text{C}+\text{R}}{6a^2}$; where . . . .

$\text{A} = \dfrac{1}{a}\tan.^{-1}\dfrac{x}{a}$, $\text{P} = x(a^2 + x^2)^{-1}$, $\text{Q} = x(a^2 + x^2)^{-2}$, and $\text{R} = x(a^2 + x^2)^{-3}$.

3. $\int\dfrac{dx\sqrt{a^2 + x^2}}{x^8} =$ . . . . . .

$$- \frac{(a^2+x^2)^{\frac{3}{2}}}{7a^2}\left\{\frac{1}{x^7} - \frac{4}{5a^2x^5} + \frac{2.4}{3.5a^4x^3}\right\}.$$

4. $\int\dfrac{dx}{x^{2n}\sqrt{1+x^2}} = -\dfrac{\sqrt{1+x^2}}{x^{2n-1}}\left\{\dfrac{1}{2n-1}\right.$ . . . . . .

$$-\frac{2n-2}{(2n-1)(2n-3)}x^2+\frac{(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}x^4-\ldots$$

$$\pm\frac{(2n-2)(2n-4)\ldots6.4}{(2n-1)(2n-3)\ldots7.5.3}x^{2n-4}\Big\}\ldots\ldots\ldots$$

$$\mp\frac{(2n-2)(2n-4)\ldots6.4.2}{(2n-1)(2n-3)\ldots7.5.3}\quad\frac{\sqrt{1+x^2}}{x}.$$

5. $\int x^{m-1}dx(1-x^2)^{\frac{2n-1}{2}}$ between $x=0$, $x=1$, $=\ldots$

$$\frac{1.3.5\cdots(2n-1)}{(m+1)(m+3)(m+5)\cdots m+2n-1}\left(\int\frac{x^{m-1}dx}{\sqrt{1-x^2}}\right).$$

6. $$\int\frac{x^m dx}{(x+\sqrt{1+x^2})^2}=\frac{2x^{m+3}}{m+3}+\frac{x^{m+1}}{m+1}-\ldots\ldots$$

$$2(1+x^2)^{\frac{3}{2}}\left\{\frac{x^m}{m+3}-\frac{mx^{m-2}}{(m+3)(m+1)}+\frac{m(m-2)x^{m-3}}{(m+3)(m+1)(m-1)}\right\}$$

$$+\frac{2m(m-2)(m-4)}{(m+3)(m+1)(m-1)}\int\sqrt{1+x^2}\,x^{m-5}dx.$$

7. $$du=\frac{xdx}{x^3+1}\therefore 3u=\sqrt{3}\tan.^{-1}.\frac{2x-1}{\sqrt{3}}-l\frac{x+1}{\sqrt{x^2-x+1}}.$$

8. $$du=\frac{x^4dx}{x^6-1}\therefore 6u=l\frac{x-1}{x+1}+l\sqrt{\frac{x^2-x+1}{x^2+x+1}}\ldots.$$

$$+\sqrt{3}\tan.^{-1}\frac{\sqrt{3}\cdot x}{1-x^2}.$$

9. $$du=\frac{dx}{(1+x)^3\sqrt{1+x+x^2}}\therefore u=\int\frac{2(1-2y)^2dy}{(y^2-2y)^3}.$$

10. $$\int x^{nx}dx=x\left\{\frac{1}{1^1}-\frac{nx}{2^2}+\frac{n^2x^2}{3^3}-\ldots\right\}$$

$$+\frac{nx^2lx}{1}\left\{\frac{1}{2^1}-\frac{nx}{3^2}+\frac{n^2x^2}{4^3}-\ldots\right\}$$

$$+\frac{n^2x^3lx^2}{1.2}\left\{\frac{1}{3^1}-\frac{nx}{4^2}+\frac{n^2x^2}{5^3}+\ldots\right\}$$

$$+\frac{n^3x^4lx^3}{1.2.3}\left\{\frac{1}{4^1}-\frac{nx}{5^2}+\frac{n^2x^2}{6^3}-\ldots\right\}$$

$$+\ldots.$$

11. $$\int\frac{dx}{(a+b\sin.x)^2}=\frac{b}{a^2-b^2}\,\frac{\cos.x}{a+b\sin.x}+\frac{a}{a^2-b^2}\int\frac{dx}{a+b\sin.x}.$$

12. $\int e^x \sin.^3x\,dx = \frac{e^x}{10}\left\{\sin.^3x + 3\cos.^3x + 3\sin.x - 6\cos.x\right\}$

13. $du = \sin(a + bx) \,.\, \sin.(m + nx)dx.$

$$\therefore u = \frac{\sin.(a-m+(b-n)x)}{2(b-n)} - \frac{\sin.(a+m+(b+n)x)}{2(b+n)}.$$

14. $du = \frac{dx}{a + b\sin.x + c\cos.x}.$

$$\therefore u = -\frac{1}{\sqrt{s^2-a^2}}\, l.\, \frac{s^2+az+\sqrt{s^2-a^2}\,\sqrt{s^2-z^2}}{a+z},$$ where

$s^2 = b^2 + c^2$ and $z = b\sin.x + c\cos.x$. If $s^2 < a^2$, u is a circular form. If $s^2 = a^2$, the form is reducible to . . . $\int \frac{dz}{a + b\sin.z}$.

15. $\int \frac{dz}{1 + n\cos.z} = \text{A}z - \text{B}\sin.z + \frac{\text{C}}{2}\sin.2z - \frac{\text{D}}{3}\sin.3z + \cdots;$

where $\text{A} = \frac{1}{\sqrt{1-n^2}}$; $\text{B} = \frac{2}{n}(\text{A} - 1)$; $\text{C} = \frac{2\text{B}}{n} - \text{A}$; . . . .

$\text{D} = \frac{2\text{C}}{n} - \text{B}$; $\text{E} = \frac{2\text{D}}{n} - \text{C}$; . . .

CHAPTER II.

Integration by Infinite Series; and between certain values of the Variable by Approximation.

1. When the known methods of integration fail, our last resource, as has been already observed, is to develope the form in an infinite series, and to integrate each term separately (vol. i. ch. 2). The kind of series in which it is desirable that the form should be developed depends upon the nature of the question. The main object is to obtain one which under the circumstances of the case shall converge: this is indispensable; and in order to diminish the labour of computation, that is to be preferred which converges the quickest. The subject of this chapter is of the greatest importance in the science of analysis, because the forms which arise in the solution of physical problems are for the most part either irrational or trancendental; and they are frequently of that complicated nature as to be integrable only by developement. There is one circumstance relating to these forms which is favourable to their integration. They are required between two given values of the variables $x = a$, $x = b$, called the *limits* of the fluent; in which case it sometimes happens that the fluent, which in its indefinite form is developable only in a series, may be expressed in finite terms. Instances have been already given in the former chapter. If $x = a$, which is the beginning of the fluent, causes it to vanish, this value of x is called the *origin* of the fluent.

The notation proposed to represent a fluent integrated between the limits b and a is $\int_b^a$, placing that limit below corresponding to which the fluent is to be subtracted. Thus, $\int_b^a x^m dx = \frac{a^{m+1} - b^{m+1}}{m+1}$ and $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$. (Encyclopædia Metropol. Art. Int. Calc. p. 832.)

These are *definite* fluents, for the correction disappears in the process of integration. The notation $\int_a^x x dx$ expresses that a either is, or is assumed to be the origin of the fluent;

and this is also a definite form; thus $\int_a^x x^m dx = \frac{x^{m+1} - a^{m+1}}{m+1}$, which does not contain the correction.

When the fluent between its limits cannot be expressed in finite or known terms, we must endeavour to approximate to its value. So many instances of developement have been given in the first volume, that in this chapter we shall add only a few examples to show that the same fluent may be expressed in different series; and shall then proceed to the theorems for approximating to the value of a fluent between its limits.

2. EXAMPLES.

Ex. 1. Required $\int (a + bx^n)^p x^{m-1} dx$ when it is integrable only by developement.

Since p is not an integer, represent the exponent by $\frac{p}{q}$; and let $du = (a + bx^n)^{\frac{p}{q}} x^{n-1} dx$.

Expand $(a + bx^n)^{\frac{p}{q}}$ by the binomial theorem; and integrating each term, we have

$$u = a^{\frac{p}{q}} \left\{ \frac{x^m}{m} + \frac{pb}{qa} \frac{x^{m+n}}{m+n} + \frac{p(p-q)b^2}{1.2q^2a^2} \frac{x^{m+2n}}{m+2n} + \cdots \right\}.$$

This series ascends by the powers of x; and it enables us to approximate to the value of u when x is small.

If a descending series be required, we have

$(a + bx^n)^{\frac{p}{q}} x^{m-1} = (b + ax^{-n})^{\frac{p}{q}} x^{\frac{m+np}{q}-1}$; and expanding as before, and integrating

$$u = qb^{\frac{p}{q}} \left\{ \frac{x^{\frac{mq+np}{q}}}{mq+np} + \frac{pa}{qb} \frac{x^{mq+n(p-q)}}{mq+n(p-q)} + \cdots \right.$$

$$\left. \frac{p(p-q)a^2}{1.2q^2b^2} \frac{x^{mq+n(p-2q)}}{mq+n(p-2q)} + \cdots \right\}.$$

If q is an even number, both the series are possible; but if q is odd, and either a or b negative, one of them contains $\sqrt{-1}$.

Ex. 2. Required the whole fluent of $\frac{dx}{\sqrt{1-x^4}}$.

We have $\frac{dx}{\sqrt{1-x^2}\sqrt{1+x^2}} = \ldots\ldots\ldots$

$$\frac{dx}{\sqrt{1-x^2}}\left\{1-\frac{1}{2}x^2+\frac{1.3}{2.4}x^4-\frac{1.3.5}{2.4.6}x^6+\cdots\right\}.$$

$\therefore$

But $\int\frac{dx}{\sqrt{1-x^2}} = \sin.^{-1}x = \text{A}$ | $(\text{A}) = \frac{\pi}{2}$

$\int\frac{x^2dx}{\sqrt{1-x^2}} = \frac{1}{2}\text{A} - \frac{x\sqrt{1-x^2}}{2} = \text{B}$ | $(\text{B}) = \frac{1}{2}\cdot\frac{\pi}{2}$

$\int\frac{x^4dx}{\sqrt{1-x^2}} = \frac{3}{4}\text{B} - \frac{x^3\sqrt{1-x^2}}{4} = \text{C}$ | $(\text{C}) = \frac{1.3}{2.4}\cdot\frac{\pi}{2}$

$\int\frac{x^6dx}{\sqrt{1-x^2}} = \frac{5}{6}\text{C} - \frac{x^5\sqrt{1-x^2}}{6} = \text{D}$ | $(\text{D}) = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}$;

. . . . = ... | . . = . .

whence $\int_0^1\frac{dx}{\sqrt{1-x^4}} = \left\{1-\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2 \ldots\ldots\right.$

$$\left.-\left(\frac{1.3.5}{2.4.6}\right)^2+\cdots\right\}\frac{\pi}{2}.$$

Similarly, it may be shown that

$$\int_0^1\frac{dx}{\sqrt{x(1-x^2)}} = \left\{1-\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2-\left(\frac{1.3.5}{2.4.6}\right)^2+\cdots\right\}\pi,$$

which is also deducible from the former by substituting $x = z^2$.

Ex. 3. Required the time of descent down a quadrant situated in a vertical plane.

Let a = rad. | then the velocity $= \sqrt{4mx}$ (Mech. p. 60)

$\frac{x}{a}$ = sin. $\angle$ described | and $dt = \frac{ds}{\text{vel.}} = \frac{1}{\sqrt{4m}}\cdot\frac{adx}{x^{\frac{1}{2}}\sqrt{a^2-x^2}}$.

Let $du = \frac{adx}{x^{\frac{1}{2}}\sqrt{a^2-x^2}} = x^{-\frac{1}{2}}dx\left\{1+\frac{1}{2}\frac{x^2}{a^2}+\frac{1.3}{2.4}\frac{x^4}{a^4}\cdots\right.$

$$\left.+\frac{1.3.5}{2.4.6}\frac{x^6}{a^6}+\frac{1.3.5.7}{2.4.6.8}\frac{x^8}{a^8}+\cdots\right\} \ldots\ldots\ldots$$

$$\therefore u = 2x^{\frac{1}{2}}+\frac{x^{\frac{5}{2}}}{5a^2}+\frac{3}{4}\frac{x^{\frac{9}{2}}}{9a^4}+\frac{3.5}{4.6}\frac{x^{\frac{13}{2}}}{13a^6}+\frac{3.5.7}{4.6.8}\frac{x^{\frac{17}{2}}}{17a^8}\cdots$$

. . . + cor.

$$\therefore (u) = a^{\frac{1}{2}}\left\{2 + \frac{1}{5} + \frac{3}{4}\times\frac{1}{9} + \frac{3.5}{4.6}\times\frac{1}{13} + \frac{3.5.7}{4.6\,8}\times\frac{1}{17} + \cdots\right\}$$

To compute the fluent by this series, let $a = 1$ foot; then

$$(u) = 2 + \frac{1}{5} + \frac{\text{A}}{9} + \frac{\text{B}}{13} + \frac{\text{C}}{17} + \frac{\text{D}}{21} + \cdots$$

where $\text{A} = \frac{3}{4}$, $\text{B} = \frac{5}{6}\text{A}$, $\text{C} = \frac{7}{8}\text{B}$, $\cdots$

This converges so slowly that we shall express the required fluent by means of another series, which, though obtained by an indirect and a more laborious process, is better suited to the purpose of computation.

Let $x = co.v.s \angle$ described; then

$$dt = \frac{1}{\sqrt{4m}}\,\frac{-dx}{\sqrt{1-x}\sqrt{2x-x^2}}.$$

Here $du = \dfrac{-dx}{\sqrt{1-x}\sqrt{2x-x^2}}$ . . . . . .

$$= \frac{1}{\sqrt{2}}\,\frac{-dx}{\sqrt{x-x^2}\sqrt{1-\frac{x}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$= -\frac{1}{\sqrt{2}}\,\frac{dx}{\sqrt{x-x^2}}\left\{1 + \frac{1}{2}\frac{x}{2} + \frac{1.3}{2.4}\frac{x^2}{4} + \frac{1.3.5}{2.4.6}\frac{x^3}{8} + \ldots\right\}.$$

But $\int \dfrac{dx}{\sqrt{x-x^2}} = v \,.\, s^{-1}2x = \text{A}$,

$$\int \frac{xdx}{\sqrt{x-x^2}} = \tfrac{1}{2}\text{A} - \sqrt{x-x^2} = \text{B},$$

$$\int \frac{x^2dx}{\sqrt{x-x^2}} = \tfrac{3}{4}\text{B} - \frac{x\sqrt{x-x^2}}{2} = \text{C},$$

$$\int \frac{x^3dx}{\sqrt{x-x^2}} = \tfrac{5}{6}\text{C} - \frac{x^2\sqrt{x-x^2}}{3} = \text{D},$$

. . . = . . .; from which the law is sufficiently manifest; and we have

$$(u) = \frac{\pi}{\sqrt{2}}\left\{1 + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{1.3}{2.4}\right)^2 + \frac{1}{8}\left(\frac{1.3.5}{2.4.6}\right)^2 + \ldots\right\}.$$

To compute by means of this series, we have

$(u) = \frac{\pi}{\sqrt{2}}\left\{1 + \frac{A}{2} + \frac{B}{2^2} + \frac{C}{2^3} + \ldots\right\}$, where $A = \frac{1}{2^2}, \ldots$

$B = \frac{3^2}{4^2}A$, $C = \frac{5^2}{6^2}B$, . . .

$\therefore$

1	$= 1.000\ 000$	1	$= 1.000\ 000$
A	$= .250\ 000$	$\frac{A}{2}$	$= .125\ 000$
B	$= .140\ 625\ 00$	$\frac{B}{4}$	$= .035\ 156\ 25$
$C = \frac{100B}{144}$	$= .097\ 656\ 25$	$\frac{C}{8}$	$= .012\ 207\ 03$
D	$= .074\ 768\ 06$	$\frac{D}{16}$	$= .004\ 673\ 00$
E	$= .060\ 562\ 12$	$\frac{E}{32}$	$= .001\ 892\ 5$
F	$= .050\ 889\ 00$	$\frac{F}{64}$	$= .000\ 795\ 1$
G	$= .043\ 878\ 11$	$\frac{G}{128}$	$= .000\ 342\ 7$
		$\frac{H}{256}$	$= .000\ 113$
			$2)1.180\ 180$
			$.590\ 09$

Whence $(t) = \frac{\pi}{\sqrt{2m}} \times .59 = .9269$.

The time of descent down the quadrant : the time down the vertical diameter : : $\frac{\pi}{\sqrt{2m}} \times .59 : \sqrt{\frac{2}{m}} : : \frac{\pi}{2} \times .59 :$ 1 : : .9267 : 1 : : 1000 : 1079 : : 100 : 108 nearly; which is the ratio deduced by T. Simpson (Flux. vol. ii. sect. 7, p. 138).

Cor. The form of Ex. 2 may be computed by means of this series; for substituting $y^2 = x$, we have, if $a = 1$, $du = -\frac{2dy}{\sqrt{1 - y^4}}$ or $\int_0^1 \frac{dy}{\sqrt{1 - y^4}} = \left(\frac{u}{2}\right) = \frac{\pi}{\sqrt{2}} \times .59009\ldots$ $= 1.310$.

Ex. 4. $dt = \dfrac{-rdz}{\sqrt{(r^2-z^2)(c+4mz)-c'^2}}$.

Laplace in considering the motion of a corpuscle in a spherical surface deduces this form, which he integrates in the following manner (Mec. Celeste, tom. i. liv. 1. 11).

Assume $(r^2-z^2)(c+4mz)-c'^2=(a-z)(z-b)(4mz+f)$, where a, b and f are determinable from the three equations

$$f = \frac{4m(r^2+ab)}{a+b},\ c = \frac{4m(r^2-a^2-ab-b^2)}{a+b},\ \ldots\ldots$$

$$c'^2 = \frac{4m(r^2-a^2)(r^2-b^2)}{a+b}.$$

Since a and b are each functions of $(c,\ c')$, they may be considered as the new arbitrary constants of the problem; and it appears from the form of the assumption that they are the limits of the fluent, of which a is the greatest, and b the least value of z.

Also assume $\sin.\theta = \sqrt{\dfrac{a-z}{a-b}}$; then $\cos.\theta = \sqrt{\dfrac{z-b}{a-b}}$; and $z = a\cos.^2\theta + b\sin.^2\theta = a - (a-b)\sin.^2\theta$ or . . . $= b + (a-b)\cos.^2\theta$; whence $dz = -2(a-b)\sin.\theta\cos.\theta d\theta$, $a - z = (a-b)\sin.^2\theta$, $z - b = (a-b)\cos.^2\theta$ and $4mz + f = 4m(a-(a-b)\sin.^2\theta) + f$; and by substitution we have

$$dt = \frac{2rd\theta}{\sqrt{4m(a-(a-b)\sin.^2\theta)+f}} \quad \ldots\ldots$$

$$= \frac{2rd\theta}{\sqrt{4m\left\{a-(a-b)\sin.^2\theta+\dfrac{r^2+ab}{a+b}\right\}}} \quad \ldots\ldots$$

$$= \frac{r\sqrt{a+b}}{\sqrt{m(a^2+2ab+r^2)}} \frac{d\theta}{\sqrt{1-\gamma^2\sin.^2\theta}}, \text{ where } \gamma^2 = \frac{a^2-b^2}{a^2+2ab+r^2}.$$

To integrate $\dfrac{d\theta}{\sqrt{1-\gamma^2\sin.^2\theta}} = du$, we have

$$du = d\theta\left\{1 + \frac{1}{2}\gamma^2\sin.^2\theta + \frac{1.3}{2.4}\gamma^4\sin.^4\theta + \frac{1.3.5}{2.4.6}\gamma^6\sin.^6\theta + \ldots\right\}$$

But from ch. 1, 47, we have

$$\int \sin.^2\theta d\theta = -\cos.\theta\frac{\sin.\theta}{2} + \frac{\theta}{2},$$

$$\int \sin.^4\theta d\theta = -\cos.\theta\left\{\frac{\sin.^3\theta}{3}+\frac{2\sin.\theta}{3.1}\right\}+\frac{3.1}{4.2}\theta,$$

$$\int \sin.^6\theta d\theta = -\cos.\theta\left\{\frac{\sin.^5\theta}{5}+\frac{4\sin.^2\theta}{5.3}+\frac{4.2\sin.\theta}{5.3.1}\right\}+\frac{5.3.1}{6.4.2}\theta,$$

. . . . = — . . . .

Now the limits are $z=a$, $z=b$; or since $z=a\cos.^2\theta+b\sin.^2\theta$, they are $\theta=0$, $\theta=\frac{\pi}{2}$, and consequently we have

$$(u)=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2\gamma^2+\left(\frac{1.3}{2.4}\right)^2\gamma^4+\left(\frac{1.3.5}{2.4.6}\right)^2\gamma^6+\ldots\right\}.$$

In the problem the origin of the co-ordinates is at the centre of the sphere, $r=$ rad., and z is the vertical co-ordinate. Let $\mathrm{T}=$ the time of $\frac{1}{2}$ an oscillation, or the time that elapses between $z=a$, $z=b$; then the result is

$$\mathrm{T}=\frac{\pi r\sqrt{a+b}}{\sqrt{4m(a^2+2ab+r^2)}}\left\{1+\left(\frac{1}{2}\right)^2\gamma^2+\left(\frac{1.3}{2.4}\right)^2\gamma^4 \ . \ . \ .\right.$$
$$\left.+\left(\frac{1.3.5}{2.4.6}\right)^2\gamma^6+\ldots\right\}.$$

If the corpuscle oscillate in a vertical plane as in ex. 3, we have $a=r$, and $\gamma^2=\frac{r-b}{2r}$ and

$$\mathrm{T}=\frac{\pi}{2}\sqrt{\frac{r}{2m}}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{r-b}{2r}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^3\left(\frac{r-b}{2r}\right)^3+\cdots\right\}.$$

If the oscillation be small or $r=b$ nearly, we have

$$2\mathrm{T}=\pi\sqrt{\frac{r}{2m}},\text{ or } 2\mathrm{T}:\sqrt{\frac{r}{2m}}::\pi:1.$$

3. *Required to obtain the value of a fluent between its limits by the method of approximation.*

Let the proposed fluxion be $du=\mathrm{x}dx$, where $u=fx$; and let it be required to find the value of u between $x=a$, $x=b$, where $b-a=h$.

By Taylor's theorem,

$$f(x+h)=fx+\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\ldots$$

$$=fx+\mathrm{x}\ \frac{h}{1}+\frac{d\mathrm{x}}{dx}\frac{h^2}{1.2}+\frac{d^2\mathrm{x}}{dx^2}\frac{h^3}{1.2.3}\cdots$$

In this equation suppose $x = a$; and represent the corresponding values of $\mathrm{X}, \frac{d\mathrm{X}}{dx}, \frac{d^2\mathrm{X}}{dx^2}, \ldots$ by $\mathrm{Y}, \mathrm{Y}', \mathrm{Y}'', \ldots$; then we have $f(a+h) = fa + \mathrm{Y}\frac{h}{1} + \mathrm{Y}'\frac{h^2}{1.2} + \mathrm{Y}''\frac{h^3}{1.2.3} \ldots$ or

$$f(a+h) - fa = fb - fa = \int_a^b \mathrm{X}dx = \mathrm{Y}\frac{h}{1} + \mathrm{Y}'\frac{h^2}{1.2} + \mathrm{Y}''\frac{h^3}{1.2.3} \ldots$$

If h be small, this series converges and an approximate value of $\int_a^b \mathrm{X}dx$ may be obtained. But in order that h may not be of a determinate magnitude, let us suppose the interval $b - a$ to be divided into n equal parts, each $= h$; then by increasing n, h may be diminished ad libitum, for $\frac{b-a}{n} = h$.

Represent as before the values of $\mathrm{X}, \frac{d\mathrm{X}}{dx}, \frac{d^2\mathrm{X}}{dx^2}, \ldots$ when $x = a$, by $\mathrm{Y}, \mathrm{Y}', \mathrm{Y}'' \ldots$; also, when $x = a + h$, by $\mathrm{Y}_1, \mathrm{Y}'_1, \mathrm{Y}''_1 \ldots$; when $x = a + 2h$, by $\mathrm{Y}_2, \mathrm{Y}'_2, \mathrm{Y}''_2, \ldots$ &c.; when $x = a + (n-1)h = b - h$, by $\mathrm{Y}_{n-1}, \mathrm{Y}'_{n-1}, \mathrm{Y}''_{n-1} \ldots$.

Also denote the corresponding values of u by $\mathrm{U}, \mathrm{U}_1, \mathrm{U}_2, \ldots \mathrm{U}_{n-1}$; then the required value of u or $\int_a^b \mathrm{X}dx$ is $\mathrm{U}_n - \mathrm{U}$.

It has been shown that

$$\mathrm{U}_1 - \mathrm{U} = \mathrm{Y}\frac{h}{1} + \mathrm{Y}'\frac{h^2}{1.2} + \mathrm{Y}''\frac{h^3}{1.2.3} + \ldots$$

$$\mathrm{U}_2 - \mathrm{U}_1 = \mathrm{Y}_1\frac{h}{1} + \mathrm{Y}'_1\frac{h^2}{1.2} + \mathrm{Y}''_1\frac{h^3}{1.2.3} + \ldots$$

$$\ldots\ldots = \ldots\ldots$$

$$\mathrm{U}_n - \mathrm{U}_{n-1} = \mathrm{Y}_{n-1}\frac{h}{1} + \mathrm{Y}'_{n-1}\frac{h^2}{1.2} + \mathrm{Y}''_{n-2}\frac{h^3}{1.2.3} + \ldots$$

Whence
$$\mathrm{U}_n - \mathrm{U} = \frac{h}{1}\left\{\mathrm{Y} + \mathrm{Y}_1 + \mathrm{Y}_2 \ldots + \mathrm{Y}_{n-1}\right\}$$
$$+ \frac{h^2}{1.2}\left\{\mathrm{Y}' + \mathrm{Y}'_1 + \mathrm{Y}'_2 \ldots + \mathrm{Y}'_{n-1}\right\}$$
$$+ \frac{h^3}{1.2.3}\left\{\mathrm{Y}'' + \mathrm{Y}''_1 + \mathrm{Y}''_2 \ldots + \mathrm{Y}''_{n-1}\right\}$$
$$+ \ldots \text{ to } n \text{ terms.}$$

By assuming n a sufficiently large number, any degree of

convergency that may be required can be given to this series.

This method fails when any of the fluxional coefficients of u between the limits become infinite.

4. In the preceding article, if we suppose the fluent to represent the area of a curve, in which case x may be called the *ordinate* of the fluent, its value is expressed in terms of the 1st, 2d, 3d, ... $n-1$th ordinates and of their fluxional coefficients. The value of the fluent may be also expressed in a series in terms of the 2d, 3d, ... $n-1$th, nth ordinates and of their fluxional coefficients.

For by Taylor's theorem, we have, taking h negative,

$$\mathrm{U}_{n-1} = \mathrm{U}_n - \frac{d\mathrm{U}_n}{dx}\frac{h}{1} + \frac{d^2\mathrm{U}_n}{dx^2}\frac{h^2}{1.2} - \ldots$$

$$= \mathrm{U}_n - \mathrm{Y}_n\frac{h}{1} + \mathrm{Y}'_n\frac{h^2}{1.2} - \mathrm{Y}''_n\frac{h^3}{1.2.3} + \ldots$$

$$\text{or } \mathrm{U}_n - \mathrm{U}_{n-1} = \mathrm{Y}_n\frac{h}{1} - \mathrm{Y}'_n\frac{h^2}{1.2} + \mathrm{Y}''_n\frac{h^3}{1.2.3} - \ldots$$

$$\text{and } \mathrm{U}_{n-1} - \mathrm{U}_{n-2} = \mathrm{Y}_{n-1}\frac{h}{1} - \mathrm{Y}'_{n-1}\frac{h^2}{1.2} + \mathrm{Y}''_{n-1}\frac{h^3}{1.2.3} - \ldots$$

$$\ldots\ldots = \ldots$$

$$\mathrm{U}_1 - \mathrm{U} = \mathrm{Y}_1\frac{h}{1} - \mathrm{Y}'_1\frac{h^2}{1.2} + \mathrm{Y}''_1\frac{h^3}{1.2.3} - \ldots$$

$$\text{Whence } \mathrm{U}_n - \mathrm{U} = \frac{h}{1}\left\{\mathrm{Y}_1 + \mathrm{Y}_2 \ldots + \mathrm{Y}_n\right\}$$

$$- \frac{h^2}{1.2}\left\{\mathrm{Y}'_1 + \mathrm{Y}'_2 \ldots + \mathrm{Y}'_n\right\}$$

$$+ \frac{h^3}{1.2.3}\left\{\mathrm{Y}''_1 + \mathrm{Y}''_2 \ldots + \mathrm{Y}''_n\right\}$$

$$- \ldots \text{ to } n \text{ terms.}$$

The advantage which this series possesses over the former is, that since its terms are alternately positive and negative, the values of $\mathrm{U}_n - \mathrm{U}$ computed by means of it are alternately greater and less than the true value; and the computer is thus enabled to ascertain to how many places of decimals the result is true.

5. Euler has also given the following approximation.

Since the first series gives a result which is too small; and the second, if we stop at a term which is positive, a result

which is too large, if we add the series together, the semi-sum will give a nearer approximation than either taken separately; thus we have

$$\int_a^b x\,dx = \frac{h}{1}\left\{ \mathrm{Y}_1 + \mathrm{Y}_2 \ldots + \mathrm{Y}_{n-1} + \tfrac{1}{2}(\mathrm{Y} + \mathrm{Y}_n) \right\}$$
$$+ \frac{h^2}{1.2\times 2}\left\{ \mathrm{Y}' - \mathrm{Y}'_n \right\}$$
$$+ \frac{h^3}{1.2.3}\left\{ \mathrm{Y}''_1 + \mathrm{Y}''_2 \ldots + \mathrm{Y}''_{n-1} + \tfrac{1}{2}(\mathrm{Y}'' + \mathrm{Y}''_n) \right\}$$
$$+ \frac{h^4}{1.2.3.4\times 2}\left\{ \mathrm{Y}''' - \mathrm{Y}'''_n \right\}$$
$$+ \&c.$$

The propositions contained in this and the two former articles are geometrically illustrated in the Princ., vol. i. lem. 2, in which the first term of the first series is the sum of the circumscribed parallelograms; the first term of the second, the sum of the inscribed parallelograms; and the first term of the third, the sum of the inscribed trapezia.

It is supposed that none of the terms which compose the coefficients of h, h^2, h^3, ... suffer a change of sign; or that none of the intermediate points of the curve offer any peculiarity.

Ex. Required to approximate to the value of $\int_0^1 e^{-\frac{1}{x}} dx$.

Here $\mathrm{x} = e^{-\frac{1}{x}}$, $\frac{d\mathrm{x}}{dx} = e^{-\frac{1}{x}}\frac{1}{x^2}$, $\frac{d^2\mathrm{x}}{dx^2} = e^{-\frac{1}{x}}\left\{\frac{1}{x^4} - \frac{2}{x^3}\right\}$,

$\frac{d^3\mathrm{x}}{dx^3} = e^{-\frac{1}{x}}\left\{\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right\}, \ldots$ $\therefore \mathrm{Y}=0$, $\mathrm{Y}'=0$, $\mathrm{Y}''=0, \cdots$

for $\frac{1}{x^n e^{-\frac{1}{x}}} = 0$ when $x = 0$; and consequently $x = 0$ is the origin of the fluent.

Divide the interval $x = 1$ into n equal parts, in which case, $nh = 1$ or $h = \frac{1}{n}$; and we have from the series, art. 5,

$$\mathrm{U}_n = \frac{1}{n}\left\{ e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + \ldots + e^{-\frac{n}{n-1}} + \tfrac{1}{2}e^{-\frac{n}{n}} \right\}$$
$$- \frac{1}{4n^2 e^{-1}}$$

$$+\frac{1}{6n^3}\left\{e^{-\frac{n}{1}}(n^4-2n^3)+e^{-\frac{n}{2}}\left(\frac{n^4}{16}-\frac{2n^3}{8}\right)+e^{-\frac{n}{3}}\left(\frac{n^4}{81}-\frac{2n^3}{27}\right)\right.$$

$$\left.\cdots+e^{-\frac{n}{n-1}}\left(\frac{n^4}{(n-1)^4}-\frac{2n^3}{(n-1)^3}\right)-\tfrac{1}{2}e^{-1}\right\}$$

$$-\frac{1}{48n^4}e^{-1}$$

$$+\ \&c.$$

If we suppose $n = 10$, and compute the required fluent by the terms here stated, the result will be true to 6 places of decimals; if we suppose $n = 20$, the result will be 20 times more accurate. (Euler, Calc. Int., vol. i. art. 327).

6. From this example the advantage of the method of approximation over the common method of developement is very obvious. For if we develope $e^{-\frac{1}{x}}dx$ in a series, the terms when integrated become infinite at $x = 0$; and we cannot obtain the required value of the fluent.

There are on the contrary other examples which, though readily integrated by the common methods, fail to be integrable by approximation in consequence of the resulting values of Υ, Υ_1, ... Thus to find the whole fluent of $\frac{dx}{\sqrt{1-x}}$, which $= -2\sqrt{1-x} + \text{cor.} = 2$, we have $\Upsilon_n = \infty$; or this method fails. See also art. 2, ex. 2. In these cases the form may be frequently changed by substitution into one which will not fail. Thus let it be required to find the whole fluent of $\frac{\text{P}dx}{(1-x)^{\frac{p}{q}}}$, where $\frac{p}{q}$ is a proper fraction.

Here $\frac{\text{P}}{(1-x)^{\frac{p}{q}}} = \infty$ when $x = 1$; but substituting $1 - x = y^q$, the form becomes $-q\text{P}y^{q-p-1}dy$; which does not fail either when $x = 0$, or $x = 1$, or at any intermediate value.

7. It may be observed in general of the method of approximation by equal intervals that, cæteris paribus, the result is the more accurate in proportion as the change of the ordinate is the less in passing from one interval to an-

other. The number of the intervals which must be taken in order to obtain a given degree of accuracy must of course depend upon the nature of the fluent: it will be the greater in proportion as the change of the ordinate is the greater. When the change is infinite, the number of the intervals should also be infinite; and this is the reason why the method fails in all such cases. If we would calculate by this method the value of the hyperbolick area whose equation is $y = \frac{a^{n+1}}{(1-x)^n}$ between $x = 0$, $x = 1$, we should fail, though when $n < 1$ the area is finite (vol. i. ch. 9, 3, ex. 5); and the reason is, that at $x = 1$ a very small change in the value of x produces an infinite change in the value of y; otherwise the area would not be finite.

It is upon these considerations that Euler proposes another method. He substitutes only for that part of the fluent's course which fails in consequence of y becoming infinite.

8. EXAMPLES.

Ex. 1. $\int_0^1 \frac{xdx}{\sqrt{1-x^3}}$.

Instead of dividing the whole value of x into intervals, each $= h$, let the last interval be ω; then to find the value of the fluent between $x=1$, $x=1-\omega$, substitute $x=1-z$; and since the limits of z are o and ω, the form, which becomes $\frac{(1-z)dz}{\sqrt{1-(1-z)^3}}$, may be considered nearly the same as $\frac{dz}{\sqrt{3z}}$, which integrated between $z=o$, $z=\omega$, is $2\sqrt{\frac{\omega}{3}}$.

If in this example the exponent of the binomial in the denominator had been greater than 1, we should have had ω in the denominator of the result, or the fluent $= \infty$; which agrees with vol. i. ch. 9, 3, ex. 5.

Ex. 2. $\int_0^1 \frac{x^2dx}{\sqrt{a^4-x^4}}$.

Here to find the value of the fluent during the last interval ω, substitute $x = a - z$; then

$$du = \frac{(a-z)^{\frac{1}{2}}dz}{\sqrt{4a^3z - 6a^2z^2 + 4az^3 - z^4}} = \frac{a^{\frac{1}{2}}dz}{2z^{\frac{1}{2}}} \frac{\left(1 - \frac{z}{a}\right)^2}{\sqrt{1 - \frac{3z}{2a} + \frac{z^2}{a^2}}} \cdots$$

$$= \frac{a^{\frac{1}{2}}dz}{2z^{\frac{1}{2}}} \frac{1 - \frac{2z}{a} + \frac{z^2}{a^2}}{1 - \frac{3z}{4a} + \frac{7z^2}{32a^2}} = \frac{a^{\frac{1}{2}}dz}{2z^{\frac{1}{2}}}\left(1 - \frac{5z}{4a} - \frac{5z^2}{32a^2}\right) \text{ and}$$

$(u) = a^{\frac{1}{2}}\omega^{\frac{1}{2}}\left\{1 - \frac{5\omega}{6a} - \frac{\omega^2}{32a^2}\right\}$ which process may be continued to as many terms as may be necessary.

Ex. 3. $du = \frac{\mathrm{P}dx}{\sqrt{(a-x)(x-b)}}$.

Here x may not be greater than a or less than b; and the ordinates at both the limits are infinite. The ordinate also during the progress of the fluent admits of a minimum value; and in consequence the proposed form cannot be conveniently integrated by the method of approximation by equal intervals, and we must have recourse to substitution.

Euler substitutes $x = \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\cos.\theta$; then $a - x = \frac{1}{2}(a-b)(1+\cos.\theta)$ and $x - b = \frac{1}{2}(a-b)(1-\cos.\theta)$ or $\sqrt{(a-x)(x-b)} = \frac{1}{2}(a-b)\sin.\theta$; whence $du = \mathrm{P}d\theta$; by which transformation the ordinate is not infinite at either limit, unless there is some peculiarity in the function P.

Generally if $du = \frac{\mathrm{P}dx}{((a-x)^p(x-b)^q)^{\frac{1}{2n}}}$ where p and q are less than $2n$; then if p and q are unequal and $p > q$, du may be put under the form $\frac{\mathrm{P}dx(x-b)^{\frac{p-q}{2n}}}{((a-x)^p(x-b)^p)^{\frac{1}{2n}}}$, and substituting as before, $u = \left(\frac{a-b}{2}\right)^{1-\frac{p+q}{2n}} \int \mathrm{P}d\theta \sin.\theta^{1-\frac{p}{n}}(1-\cos.\theta)^{\frac{p-q}{2n}}$.

No general rules can be given to direct the student in the choice of the transformation which may be the best suited to any proposed case: this is a kind of knowledge which cannot be taught in an elementary treatise; it is to be acquired only by a diligent perusal of the best authors.

Euler in the Calc. Int. ch. 8, has given the following theorem, which is sometimes useful.

9. *The fluent of* $\frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}}$ *from* $x = 0$ *to* $x = 1$ *is equal to the fluent of* $\frac{x^{q-1}dx}{(1-x^n)^{\frac{n-p}{n}}}$ *from* $x = 1$ *to* $x = 0$.

For substitute $1 - x^n = y^n$; then when $x = 0$, $y = 1$, and when $x = 1$, $y = 0$.

Also $(1 - x^n)^{\frac{n-q}{n}} = y^{n-q}$; and since $x^p = (1 - y^n)^{\frac{p}{n}}$, we have $x^{p-1}dx = -(1 - y^n)^{\frac{p-n}{n}} y^{n-1}dy$; wherefore

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = -\int \frac{(1-y^n)^{\frac{p-n}{n}} y^{n-1}dy}{y^{n-q}} = -\int \frac{y^{q-1}dy}{(1-y^n)^{\frac{n-p}{n}}};$$

and since the values of y are to be taken the same as those of x, it results that $\int_0^1 \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = \int_1^0 \frac{x^{q-1}dx}{(1-x^n)^{\frac{n-p}{n}}}$. (Legendre, Exercises du Calc. Int. p. 222).

Cor. 1. These forms may be rendered rational when $p = n - q$.

For substitute $z^n = \frac{x^n}{1-x^n}$; then $1 + z^n = \frac{1}{1-x^n}$ and $1 - x^n = \frac{1}{1+z^n}$ or $x^n = \frac{z^n}{1+z^n}$; wherefore

$$\frac{dx}{x} = \frac{dz}{z} - \frac{z^{n-1}dz}{1+z^n} = \frac{dz}{z(1+z^n)}.$$

But $du = \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = \frac{dx}{x} \cdot \left(\frac{x^n}{1-x^n}\right)^{\frac{n-q}{n}} = \frac{z^{n-q-1}dz}{1+z^n}$, a rational form.

Cor. 2. The fluent of $\frac{z^{n-q-1}dz}{1+z^n}$ from $z = 0$ to $z = \infty$ is equal to the fluent of $\frac{z^{q-1}dz}{1+z^n}$.

For when $x = 0$, $z = 0$, and when $x = 1$, $z = \infty$.

10. $\int_0^\infty \frac{x^{m-1}dx}{1+x^n} = \frac{\pi}{n\sin.\frac{m\pi}{n}}$.

First to find the sum of the logarithmick parts (ch. 1. 9), we have $-\frac{h}{n}l(1-2ax+x^2) = (\text{when } x=\infty) - \frac{2h}{n}lx$; wherefore the required sum $= -\frac{2lx}{n}(\cos.\alpha + \cos.3\alpha + \cdots \cos.5\alpha \cdots + \cos.k\alpha)$ where $\alpha = \frac{m\pi}{n}$.

Let $s = \cos.\alpha + \cos.3\alpha \cdots + \cos.k\alpha$; then
$2\sin.\alpha.s = 2\sin.\alpha\cos.\alpha + 2\sin.\alpha\cos.3\alpha \cdots + 2\sin.\alpha\cos.k\alpha$.

But (Trig. p. 27) $2\sin.\alpha\cos.n\alpha = \sin.(n+1)\alpha - \sin.(n-1)\alpha$; wherefore

$$2\sin.\alpha.s = \left.\begin{array}{l}\sin.2\alpha + \sin.4\alpha + \sin.6\alpha \cdots + \sin.(k+1)\alpha \\ -\sin.2\alpha - \sin.4\alpha - \sin.6\alpha - \cdots\end{array}\right\}$$

$$= \sin.(k+1)\alpha; \text{ or } s = \frac{\sin.(k+1)\alpha}{2\sin.\alpha}.$$

When n is *even*, $k = n-1$ or $k+1 = n$ and $s = \frac{\sin.n\alpha}{2\sin.\alpha}$
$= \frac{\sin.m\pi}{2\sin.\frac{m\pi}{n}} = 0$, since m is an integer.

When n is *odd*, $k = n-2$ and $s = \frac{\sin.(n-1)\alpha}{2\sin.\alpha} \cdots$
$= \frac{\sin.\left(m\pi - \frac{m\pi}{n}\right)}{2\sin.\frac{m\pi}{n}} = \pm\frac{1}{2}$, according as m is odd or even, and the sum $= \mp\frac{lx}{n}$. But in this case another logarithmick part is to be added, viz. $\frac{(-1)^{m-1}}{n}l(1+x) = \text{when } x=\infty$, $\frac{(-1)^{m-1}}{n}lx = \pm\frac{lx}{n}$, according as m is odd or even; and consequently the whole sum of the logarithmick parts, as before, $= 0$.

Next to find the sum of the circular parts; the first part $= \frac{2h'}{n}\tan.^{-1}\frac{d'x}{1-ax} =$, when $x = \infty$,

$\frac{2h'}{n}\tan.^{-1}\frac{-a'}{a} = \frac{2}{n}\left(\pi - \frac{\pi}{n}\right)\sin.\frac{m\pi}{n}$.

Similarly, the second part $= \frac{2}{n}\left(\pi - \frac{3\pi}{n}\right)\sin.\frac{3m\pi}{n}$, and so on; and consequently the whole sum

$$\left.\begin{aligned} &= \frac{2\pi}{n}(\sin.\alpha + \sin.3\alpha + \sin.5\alpha \cdots + \sin.k\alpha) \\ &- \frac{2\pi}{n^2}(\sin.\alpha + 3\sin.3\alpha + 5\sin.5\alpha \cdots + k\sin.k\alpha.) \end{aligned}\right\}$$

To sum the second series, we have

$$s = \cos.\alpha + \cos.3\alpha + \cos.5\alpha \cdots + \cos.k\alpha = \frac{\sin.(k+1)\alpha}{2\sin.\alpha};$$

and consequently differentiating and dividing by $d\alpha$, we have, $-(\sin.\alpha + 3\sin.3\alpha + 5\sin.5\alpha \cdots + k\sin.k\alpha) \cdots$

$$= \frac{(k+1)\cos.(k+1)\alpha}{2\sin.\alpha} - \frac{\sin.(k+1)\alpha\cos.\alpha}{2\sin.^2\alpha} \cdots$$

$$= \frac{k}{2}\cdot\frac{\cos.(k+1)\alpha}{\sin.\alpha} - \frac{\sin.k\alpha}{2\sin.^2\alpha}.$$

To sum the first, let

$s = \sin.\alpha + \sin.3\alpha \cdots + \sin.k\alpha$; then

$2\sin.\alpha \,.\, s = 2\sin.\alpha\sin.\alpha + 2\sin.\alpha\sin.3\alpha \cdots + 2\sin.\alpha\sin.k\alpha.$

But $2\sin.\alpha\sin.n\alpha = \cos.(n-1)\alpha - \cos.(n+1)\alpha$; wherefore

$$\begin{aligned} 2\sin.\alpha \,.\, s &= 1 + \cos 2\alpha + \cos.4\alpha \cdots + \cos.(k-1)\alpha \\ &\quad - \cos.2\alpha - \cos.4\alpha \cdots - \cos.(k-1)\alpha - \cos.(k+1)\alpha \end{aligned}$$

$$= v\,.\,s.(k+1)\alpha, \text{ or } s = \frac{v.s.(k+1)\alpha}{2\sin.\alpha}.$$

Hence the sum of the circular parts

$$= \frac{\pi}{n}\,\frac{v.s.(k+1)\alpha}{\sin.\alpha} + \frac{\pi}{n^2}\left\{\frac{k\cos.(k+1)\alpha}{\sin.\alpha} - \frac{\sin.k\alpha}{\sin.^2\alpha}\right\}.$$

When n is even, $k+1 = n$ and the required sum

$$= \frac{\pi}{n}\,\frac{v.s.m\pi}{\sin.\alpha} + \frac{\pi}{n^2}\left\{\frac{(n-1)\cos.m\pi}{\sin.\alpha} - \frac{\sin.(m\pi-\alpha)}{\sin.^2\alpha}\right\}$$

$$= \frac{\pi}{n\sin.\frac{m\pi}{n}}, \text{ whether } m \text{ be odd or even.}$$

When n is odd, $k = n - 2$, and the required sum

$$= \frac{\pi}{n} \cdot \frac{v.s.\cdot(m\pi - \alpha)}{\sin.\alpha} + \frac{\pi}{n^2}\left\{\frac{(n-2)\cos.(m\pi-\alpha)}{\sin.\alpha} - \frac{\sin.(m\pi - 2\alpha)}{\sin.^2\alpha}\right\}$$

$$= \frac{\pi}{n} \cdot \frac{1}{\sin.\alpha} - \frac{\pi}{n^2}\left\{\frac{2\cos.(m\pi-\alpha)}{\sin.\alpha} + \frac{\sin.(m\pi-2\alpha)}{\sin.^2\alpha}\right\}$$

$$= \frac{\pi}{n\sin.\frac{m\pi}{n}}, \text{ whether } m \text{ be odd or even.}$$

Cor. 1. $\int_0^\infty \frac{z^{n-q-1}dz}{1+z^n} = \int_0^\infty \frac{z^{q-1}dz}{1+z^n} = \frac{\pi}{n\sin.\frac{q\pi}{n}}$.

Cor. 2. $\int_0^1 \frac{x^{n-q-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = \int_0^1 \frac{x^{q-1}dx}{(1-x^n)^{\frac{q}{n}}} = \frac{\pi}{n\sin.\frac{q\pi}{n}}$.

Cor. 3. Since $\frac{1}{(1-x^n)^{\frac{q}{n}}} = 1 + \frac{q}{n}x^n + \frac{q(q+n)}{n.2n}x^{2n} \ldots + \frac{q(q+n)(q+2n)}{n.2n.3n}x^{3n}$ + &c.; we have, multiplying by $x^{q-1}dx$, and integrating each term from $x = 0$, to $x = 1$,

$$\frac{\pi}{n\sin.\frac{q\pi}{n}} = \frac{1}{q} + \frac{q}{n(q+n)} + \frac{q(q+n)}{n.2n(q+2n)} + \frac{q(q+n)(q+2n)}{n.2n.3n(q+2n)} + \cdots$$

Cor. 4. For q substitute $n - q$, and we also have

$$\frac{\pi}{n\sin.\frac{q\pi}{n}} = \frac{1}{n-q} + \frac{n-q}{n(2n-q)} + \frac{(n-q)(2n-q)}{n.2n(3n-q)} \ldots$$

$$+ \frac{(n-q)(2n-q)(3n-q)}{n.2n.3n(4n-q)} + \ldots$$

Ex. $\int_0^\infty \frac{dz}{1+z^2} = \frac{\pi}{2\sin.\frac{\pi}{2}} = \frac{\pi}{2}$.

11. In the preceding articles methods of approximation have been given by means of which fluents may be computed to any degree of accuracy that may be required. It is further useful to the computer to know the amount of the error; or rather to be able to find a limit which it cannot

exceed. A knowledge of this limit will prevent him from carrying on his calculations to an unnecessary degree of accuracy. The following Lemma must be premised.

12. *Lemma. The whole value of a function between limits has the same sign as its first fluxional coefficient at the inferior limit, provided that the first fluxional coefficient neither become infinite nor change its sign during the interval between the limits.*

Let a be the least, b the greatest value of x in $u = \int \mathrm{x} dx$; then shall $\int_a^b \mathrm{x} dx$ have the same sign + or − as the value of x has at a.

For in the series which expresses the value of $\mathrm{U}_n - \mathrm{U}$ (art. 3), h may be taken so small that the first term may be greater than the sum of the remaining terms (vol. i. ch. 3, 2); and consequently, so far as relates to their signs, we have

$\mathrm{U}_n - \mathrm{U} = \frac{h}{1}\left\{\mathrm{Y} + \mathrm{Y}_1 + \mathrm{Y}_2 \cdots + \mathrm{Y}_{n-1}\right\}$; which, since h is positive, and Y, Y_1, $\cdots \mathrm{Y}_{n-1}$, have all the same sign, is + or − according as Y is + or −.

Cor. 1. If the inferior limit be the origin of the function, then its value at any point during the interval has the same sign as its first fluxional coefficient at the inferior limit.

For, as before, $\mathrm{U}_1 - \mathrm{U} = \frac{h}{1} . \mathrm{Y}$; $\mathrm{U}_2 - \mathrm{U}_1 = \frac{h}{1} . \mathrm{Y}_1$; $\cdots$ $\mathrm{U}_3 - \mathrm{U}_2 = h_1 . \mathrm{Y}_2$; $\cdots$ but here $\mathrm{U} = 0$; wherefore $\mathrm{U}_1 = \frac{h}{1}\mathrm{Y}$; $\mathrm{U}_2 = \mathrm{U}_1 + \frac{h}{1} . \mathrm{Y}_1 = \frac{h}{1}\left\{\mathrm{Y} + \mathrm{Y}_1\right\}$, $\mathrm{U}_3 = \frac{h}{1}\left\{\mathrm{Y} + \mathrm{Y}_1 + \mathrm{Y}_2\right\} \cdots$; which all have the same sign as Y.

Cor. 2. If b be negative, $\mathrm{U}_n - \mathrm{U}$ and Y have contrary signs.

If any of the higher fluxional coefficients become infinite, this will not affect the truth of the lemma: because the terms which are neglected contain higher powers of h than the first.

It is a necessary restriction that none of the first fluxional coefficients shall be infinite; because an infinite magnitude, when it changes its sign in passing through infinity, cannot be said to be either positive or negative. The symbol ∞ in this case, like that of 0, merely denotes the intermediate

state between a positive and a negative quantity. Even when the magnitude does not change its sign, this restriction is necessary; and the reason is, that if any of the quantities Y, Y_1, $\cdots$ Y_{n-1} be infinite, their fluxional coefficients Y', Y'_1, $\cdots$ Y'_{n-1} will be infinites of the second order, and when multiplied into h^2, the product may not be neglected when compared with $\frac{h}{1}\left\{ Y + Y_1 \cdots + Y_{n-1} \right\}$ *.

13. *A function is developed by Taylor's Theorem, and all the terms neglected after the nth; required to find limits to the error of the result.*

Let $U = fx$; and when x becomes $x + h$, let U become U_1; then $U_1 = f(x + h)$; and consequently we have generally $\frac{d^n U_1}{dx^n} = \frac{d^n U_1}{dh^n}$ (vol. i. 4, 3).

Suppose h to increase from 0 to some determinate value of h; and let M and m be the greatest and least values of $\frac{d^n U_1}{dh^n} = \frac{d^n U_1}{dx^n}$ during the interval; then M and m are expressed in terms of x, which is considered as constant, and of the determinate value of h.

Now during the interval, $M - \frac{d^n U_1}{dh^n}$ and $\frac{d^n U_1}{dh^n} - m$ are always positive. But these quantities are the first fluxional

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* For further details on this important branch of the subject, I must refer the reader to Lagrange's Calcul des Fonctions, Leçon 9. Here, as in other parts of his writings, Lagrange shows an ignorance of the Newtonian doctrine of limiting ratios which was not to have been expected in so learned an author. He takes the example $u = \frac{1}{a-x} - \frac{1}{a}$: the origin of this is $x=0$; its first fluxional coefficient is $\frac{1}{(a-x)^2}$, which is infinite at $x=a$, but is positive for all values of x; and yet when $x > a$, the function u becomes negative. Lagrange asserts that the common principles of the calculus fail in explaining this result—"Ainsi les principes du Calcul Differentiel sont en defaut dans ce cas," p. 93.

coefficients respectively of $Mh - \frac{d^{n-1}U_1}{dh^{n-1}}$ and $\frac{d^{n-1}U_1}{dh^{n-1}} - mh$; or of $Mh - \left\{\frac{d^{n-1}U_1}{dh^{n-1}} - \frac{d^{n-1}U}{dx^{n-1}}\right\}$ and $\frac{d^{n-1}U_1}{dh^{n-1}} - \frac{d^{n-1}U}{dx^{n-1}} - mh$; for M and m are determinate magnitudes, and h not being contained in U, $\frac{d^{n-1}U}{dx^{n-1}}$ is a constant. These functions vanish at $h = 0$, because at that point we have $U_1 = U$ and $\frac{d^{n-1}U_1}{dh^{n-1}} = \frac{d^{n-1}U}{dx^{n-1}}$; wherefore, by the lemma, cor. 2, they have always the same sign during the limits of h; and this sign is positive, because their first fluxional coefficients are positive.

Again these functions may be shown to be the first fluxional coefficients of

$$M \cdot \frac{h^2}{1.2} - \left\{\frac{d^{n-2}U_1}{dh^{n-2}} - \frac{d^{n-2}U}{dx^{n-2}} - \frac{d^{n-1}U}{dx^{n-1}} \cdot \frac{h}{1}\right\}$$

and $\frac{d^{n-2}U_1}{dh^{n-2}} - \frac{d^{n-2}U}{dx^{n-2}} - \frac{d^{n-1}U}{dx^{n-1}} \cdot \frac{h}{1} - m \cdot \frac{h^2}{1.2}$ } and these vanish at $h = 0$;

consequently, as before, they are always positive.

By means of n similar processes, it may be shown that the two following functions are always positive during the limits of h.

$$M \cdot \frac{h^n}{1.2 \cdots n} - \left\{U_1 - U - \frac{dU}{dx}\frac{h}{1} - \frac{d^2U}{dx^2}\frac{h^2}{1.2} \cdots - \frac{d^{n-1}U}{dx^{n-1}} \cdot \frac{h^{n-1}}{1.2 \cdots (n-1)}\right\}$$

and $U_1 - U - \frac{dU}{dx} \cdot \frac{h}{1} - \frac{d^2U}{dx^2} \cdot \frac{h^2}{1.2} \cdots - \frac{d^{n-1}U}{dx^{n-1}} \cdot \frac{h^{n-1}}{1.2(n-1)} \cdots - m \cdot \frac{h^n}{1.2 \ldots n}$.

But $U_1 = U + \frac{dU}{dx}\frac{h}{1} \cdots + \frac{d^{n-1}U}{dx^{n-1}} \cdot \frac{h^{n-1}}{1.2 \cdots (n-1)} \cdots + \frac{d^nU}{dx^n} \cdot \frac{h^n}{1.2 \cdots n} + \cdots$

Whence by substitution the two following functions are always positive,

$$M \cdot \frac{h^n}{1.2 \cdots n} - \left\{\frac{d^nU}{dx^n} \cdot \frac{h^n}{1.2 \cdots n} + \frac{d^{n+1}U}{dx^{n+1}} \cdot \frac{h^{n+1}}{1.2 \cdots (n+1)} \cdots\right\}$$

and $\frac{d^n \mathrm{U}}{dx^n} \cdot \frac{h^n}{1.2\cdots n} + \frac{d^{n+1}\mathrm{U}}{dx^{n+1}} \cdot \frac{h^{n+1}}{1.2\cdots n+1} + \cdots - m \cdot \frac{h^n}{1.2\cdots n}$;

and consequently, $\mathrm{M} \cdot \frac{h^n}{1.2\cdots n}$ and $m \cdot \frac{h^n}{1.2\cdots n}$ are limits; the one greater, and the other less than the error of the result.

Cor. 1. If any two magnitudes be taken, the one greater than M and the other less than m, these à fortiori are also limits to the error of the result.

Cor. 2. By a similar process the limits may be found when U is a function of two or more variables.

Ex. 1. $\mathrm{U} = x^r$.

Here $\mathrm{U}_1 = (x + h)^r \therefore \frac{d\mathrm{U}_1}{dx} = r(x + h)^{r-1}$; . . . . . .

$\frac{d^2\mathrm{U}_1}{dx^2} = r(r-1)(x+h)^{r-2} \cdots \frac{d^n\mathrm{U}_1}{dx^n} =$ . . . . . .

$r(r-1)\cdots(r-n+1)(x+h)^{r-n}$; which are all positive.

Now $\mathrm{M} = r(r-1)\cdots(r-n+1)(x+h)^{r-n}$, and $m = r(r-1)\cdots(r-n+1)x^{r-n} \therefore$ the required limits are $\frac{r(r-1)\cdots(r-n+1)}{1.2\cdots n}(x+h)^{r-n}h^n$ . . . . . .

and $\frac{r(r-1)\cdots(r-n+1)}{1.2\cdots n}x^{r-n}h^n$; or the value of $(x + h)^r$ lies between

$$x^r + rx^{r-1}h + r \cdot \frac{r-1}{2}x^{r-2}h^2 \cdots + \frac{r(r-1)\cdots(r-n+1)}{1.2\cdots n}(x+h)^{r-n}h^n,$$

and $x^r + rx^{r-1}h + r \cdot \frac{r-1}{2}x^{r-2}h^2 \cdots + \frac{r(r-1)\cdots(r-n+1)}{1.2\cdots n}x^{r-n}h^n$.

Ex. 2. $\mathrm{U} = lx$ or $\mathrm{U}_1 = l(x + h)$.

The first fluxional coefficient is $\frac{1}{x+h}$, which provided that $x + h$ is not $= 0$, neither becomes infinite nor changes its sign. Also $\frac{d^n\mathrm{U}_1}{dx^n} = \pm\frac{1.2.3\cdots n-1}{(x+h)^n}$, according as n is odd or even; whence, if n be odd, $\mathrm{M} = \frac{1.2.3\cdots(n-1)}{x^n}$ and

$m = \dfrac{1.2.3\cdots(n-1)}{(x+h)^n}$; or $l(x+h)$ is between

$$lx + \frac{h}{x} - \frac{h^2}{2x^2} + \cdots \pm \frac{h^n}{nx^n},$$

and $lx + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \cdots \pm \dfrac{h^n}{n(x+h)^n}$.

Ex. 3. $u_1 = \text{sin.}(x+h)$.

Here $\dfrac{d^n u_1}{dx^n} = \pm\text{sin.}(x+h)$ or $\pm\text{ cos.}(x+h)$, according as n is of the form $4i$, $4i+2$ or $4i+1$, $4i+3$, which include all the possible cases.

Now $+1$ and -1 are evidently limits in all the four cases (cor. 1); whence we have $\text{sin.}(x+h)$, where $x+h$ is not $> \dfrac{\pi}{2}$, always between

$$\text{sin.}x + \text{cos.}x\frac{h}{1} - \text{sin.}x\frac{h^2}{1.2} - \text{cos.}x\frac{h^3}{1.2.3} + \cdots \pm \frac{h^n}{1.2\cdots n}.$$

Cor. 1. If $x = 0$, we have $\text{sin.}h$ always between

$$h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \cdots \pm \frac{h^n}{1.2\cdots n}.$$

Cor. 2. If $x = \dfrac{\pi}{2}$, we have $\text{cos.}h$ always between

$$1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \cdots \pm \frac{h^n}{1.2\cdots n}.$$

In these series n is greater by unity than the number of the term at which the series stops.

14. MISCELLANEOUS PRAXIS.

1. $\int \dfrac{dx}{\sqrt{1+x^2}} = l(x+\sqrt{1+x^2}) = x - \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\dfrac{x^5}{5} \cdots - \dfrac{1.3.5}{2.4.6}\dfrac{x^7}{7} + \cdots$

2. $\int \dfrac{dx}{lx} = l^2x + \dfrac{lx}{1} + \dfrac{1}{1.2}\dfrac{lx^2}{2} + \dfrac{1}{1.2.3}\dfrac{lx^3}{3} + \cdots$

3. $\int \dfrac{a^x dx}{1-x} = \dfrac{a^x}{(1-x)la}\left\{1 - \dfrac{1}{(1-x)la} + \dfrac{1.2}{(1-x)^2la^2} \cdots - \dfrac{1.2.3}{(1-x)^3la^3} + \cdots\right\}$

or $= x + \left\{1 + \frac{la}{1}\right\}\frac{x^2}{2} + \left\{1 + \frac{la}{1} + \frac{la^2}{1.2}\right\}\frac{x^3}{3} \cdots$

$$+ \left\{1 + \frac{la}{1} + \frac{la^2}{1.2} + \frac{la^3}{1.2.3}\right\}\frac{x^4}{4} \cdots$$

4. $\int_0^1 \frac{x^n dx}{\sqrt{x - x^2}} = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots n}\pi.$

5. $\int_0^{2a} \frac{dx}{\sqrt{b-x}\sqrt{2ax - x^2}} = \frac{\pi}{\sqrt{b}}\left\{1 + \frac{2a}{2^2 b} + \frac{3^2.(2a)^2}{2^2.4^2 b^2} \cdots\right.$

$$\left. + \frac{3^2.5^2(2a)^3}{2^2.4^2.6^2 b^3} + \cdots\right\}$$

6. $\int_0^{\pi} e^{2A\cos.x} dx = \pi\left\{1 + (A)^2 + \left(\frac{A^2}{1.2}\right)^2 + \left(\frac{A^3}{1.2.3}\right)^2 + \cdots\right\}$

7. $\int_0^1 \frac{lx.dx}{\sqrt{1-x^2}} = -\left\{1 + \frac{1}{2.3^2} + \frac{1.3}{2.4.5^2} + \frac{1.3.5}{2.4.6.7^2} + \cdots\right\}$

8. $\int_0^{\infty} \frac{dz}{1+z^3} = \int_0^1 \frac{dx}{(1-x^3)^{\frac{1}{3}}} = \int_0^1 \frac{xdx}{(1-x^3)^{\frac{2}{3}}} = \frac{2\pi}{3\sqrt{3}}.$

9. $\int_0^{\infty} \frac{dz}{1+z^4} = \int_0^{\infty} \frac{z^2 dz}{1+z^4} = \int_0^1 \frac{dx}{(1-x^4)^{\frac{1}{4}}} = \int_0^1 \frac{x^2 dx}{(1-x^4)^{\frac{3}{4}}} = \frac{\pi}{2\sqrt{2}}$

10. $\int_0^{\infty} \frac{dz}{1+z^6} = \int_0^{\infty} \frac{z^4 dz}{1+z^6} = \int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \int_0^1 \frac{x^4 dx}{(1-x^6)^{\frac{5}{6}}} = \frac{\pi}{3}.$

11. $\int_0^{\mathrm{T}} e^{-t^2} dt = \mathrm{T} - \frac{\mathrm{T}^3}{3} + \frac{1}{1.2}\frac{\mathrm{T}^5}{5} - \frac{1}{1.2.3}\frac{\mathrm{T}^7}{7} + \cdots$

$$\text{also} = e^{-\mathrm{T}^2}.\mathrm{T}\left\{1 + \frac{2\mathrm{T}^2}{1.3} + \frac{(2\mathrm{T}^2)^2}{1.3.5} + \frac{(2\mathrm{T}^2)^3}{1.3.5.7} + \cdots\right\}$$

$$\int_t^{\infty} e^{-t^2} dt = \frac{e^{-\mathrm{T}^2}}{2\mathrm{T}}\left\{1 - \frac{1}{2\mathrm{T}^2} + \frac{1.3}{2^2\mathrm{T}^4} - \frac{1.3.5}{2^3\mathrm{T}^6} + \cdots\right\}$$

12. $\frac{dy}{dx} - 1 + xy \therefore y = 1 + \frac{1}{1.3} + \frac{1}{1.3.5} + \frac{1}{1.3.5.7} + \cdots$

between $x = 0, x = 1.$

13. $\int_0^a \frac{dx}{\sqrt{x}\sqrt{1-x^2}} =$ the fluent of the same fluxion between

$x = \dfrac{1-a}{1+a}$, $x = 1$, where a may be any value of x.

14. The function a^h is always between

$$1 + la \cdot \frac{h}{1} + la^2 \cdot \frac{h^2}{1.2} \ldots + la^n \cdot \frac{h^n}{1.2 \ldots n},$$

and $1 + la \cdot \dfrac{h}{1} + la^2 \cdot \dfrac{h^2}{1.2} \ldots + la^n \cdot \dfrac{h^n a^h}{1.2 \ldots n}$.

CHAPTER III.

FLUXIONAL EQUATIONS.

The Theory of Arbitrary Constants.

1. *Def.* 1. THE *order* of a fluxional equation is determined by the highest fluxional coefficient which it contains.

Def. 2. The *degree* of a fluxional equation is determined by the highest power of the coefficient which marks the order; the equation being clear of radicals.

Thus, $p\sqrt{y^2 + x^2} - b - yp - x = 0$ is of the first order and second degree.

Def. 3. Equations in which y and all the fluxional coefficients of $y = fx$ are of the first degree are said to be *linear* with respect to y, because the fluxional coefficients are of the same dimensions as a right line.

2. *The genesis of fluxional equations.*

It has been shown, vol. i. ch. 4, art. 27, that the successive fluxions of $u = 0$ are also equal to nothing; and consequently all equations which result from their combination obtain at the same time, or are *simultaneous*.

Let $u = 0$ be an equation between x and y involving n constants $a, b, c, \ldots$; then differentiating on the supposition that x is the principal variable, we may by means of the resulting equation, $u' = 0$, eliminate one of the constants as a, and obtain what is called a *first derivative* of the primitive $u = 0$; if we had eliminated b or c or . . . instead of a, we might obtain n different first derivatives.

By again differentiating and eliminating, we may obtain *second derivatives*, in which two constants have disappeared; and the number of these will manifestly equal the number of combinations taken two and two in n things or $n \cdot \frac{n-1}{2}$. Alg. 230.

Similarly, the number of derivatives of the rth order, or in which r constants have disappeared

$$= \frac{n.(n-1)....(n-r+1)}{1.2 \qquad r}.$$

Cor. 1. There are n derivatives of the $n-1$th order, in which only one constant remains; and one nth derivative from which all the constants have disappeared.

Cor. 2. A derivative of the nth order has n first primitives, each involving one arbitrary constant; $n \cdot \frac{n-1}{2}$ second primitives, each involving two arbitrary constants, and so on; and one original primitive or fluent involving n arbitrary constants.

3. If the constant disappears by differentiation solely without elimination, the equation is said to be derived *immediately* from its primitive; thus, the immediate first derivative of $x^2 - 2ay - a^2 - b = 0$ is $2xdx - 2ady = 0$.

All such derivatives are said to be the *exact fluxions* of their primitives.

There are primitives which do not contain one constant more than their derivatives; these must be considered as *incomplete fluents*, or as particular cases of a canonical equation. When we retrace our steps, and from the derivative return to the primitive, it is always supposed that the latter has one constant more than the former. The value of this constant is to be determined from the conditions of the problem which has given rise to the fluxional equation.

Cor. It is manifest that the immediate derivatives, whatever be their order, are all of the first degree. If then an equation is of a higher degree than the first, we know that it is not an immediate derivative, but that it has been derived from its original primitive by differentiation combined with elimination.

4. There are other fluxional equations which may be deduced from the same primitive. This will appear in the course of the chapter. We shall not call them *derivatives*, but shall appropriate that term to such equations as are derived from their primitive by the elimination of a constant.

5. When it is said that an equation of the nth order can have only n first primitives, it is to be understood that these primitives are all *independent* equations. Garnier produces the following example, as if to show that the number of the first primitives may exceed the order of the equation *.

* It does not appear from the context what Garnier's object is

Ex. $(1-x)d^3y - 3dxd^2y = 0$.

It may be shown that this equation has the following 5 first primitives, each of which contains a different constant.

$(1-x)d^2y - 2dxdy + 2cdx^2 = 0$. (1)

$x(1-x)d^2y - (x+1)dxdy + (y-c')dx^2 = 0$. (2)

$x^2(x-1)d^2y + 2xdxdy - 2(y-c'')dx^2 = 0$. (3)

$(1-x)^3d^2y + c'''dx^2 = 0$. (4)

$(1-x)^2d^2y - (1-x)dxdy - (y-c^{iv})dx^2 = 0$. (5)

But of these only three are really independent; for adding twice the equation (2) to the difference between (3) and (1), and substituting $c''' = 2(c'' - c' - c)$, there results the equation (4); also subtracting (2) from (1), and substituting $c^{iv} = 2c + c'$, there results equation (5).

6. A derivative may be derived from its primitive in two different ways; either by differentiating the primitive, and then eliminating the constant; or by solving the primitive with respect to the constant, and then differentiating.

It is manifest that the resulting equations, though they appear under different forms, must be essentially the same; since the same constant has disappeared.

Ex. $x^2 - 2ay - a^2 - b = 0$.

Differentiating, $2x - 2ap = 0 \therefore a = \frac{x}{p}$; which substituted, gives $x^2 - \frac{2xy}{p} - \frac{x^2}{p^2} - b = 0$, or $(x^2-b)p^2 - 2xyp - x^2 = 0$. ($\alpha$.)

Next, solve the equation with respect to a, and there results $a = -y \pm \sqrt{y^2+x^2-b}$; and differentiating,

$0 = -p + \frac{yp+x}{\sqrt{y^2+x^2-b}}$ or $p\sqrt{y^2+x^2-b} - yp - x \cdots = 0$. ($\beta$.)

The equations (α) and (β) may be shown to be the same by solving (α) with respect to p, and reducing it.

7. The degree to which the derivative rises is equal to the number of dimensions of the constant which has been eliminated.

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in producing this example; he prefaces it with "Nous ferons ici une remarque importante qui n'a pas échappé à M. Dubourguet, qui l'a presentée avec beaucoup de développement dans son *Calcul Intégral*, p. 272."

The second method of deducing the derivative has this advantage over the first, that the resulting equation is solved with respect to p.

If the primitive contain n constants, by adopting the second method we may obtain a succession of derivatives of different orders, in each of which the highest fluxional coefficient rises to only one dimension.

Cor. If we eliminate x or y from two derivatives of the same order, the resulting equation, which contains one constant more than either of the derivatives, shall be of the same order, but its degree shall equal the degree of the variable which is eliminated.

8. If we eliminate p by combining any two of the n first derivatives, there will result an equation involving the n constants, which is therefore the original primitive.

9. *An equation which is homogeneous with respect to x and y, does not admit of a second derivative.*

For when the equation is homogeneous, and of n dimensions, dividing by x^n, it takes the form of a function of $\frac{y}{x}$, and consequently the values of $\frac{y}{x}$ may be found in known terms by the theory of equations. Let m be one of them; then $\frac{y}{x} = m$ or $y = mx$, therefore $p = m$, a constant quantity; and the equation cannot be differentiated a second time.

10. If from a proposed equation of the first order there be derived one of the second, either immediately or by means of elimination; and from this latter, there be deduced by any process whatever a first primitive essentially different from the proposed equation, eliminating p from the two equations of the first order, there will result the required primitive involving one arbitrary constant.

Generally, if from a proposed equation of the first order, there be deduced, immediately or not, a succession of equations of every order as far as the rth; and from this latter, there can be deduced one of the $r - 1$th order involving one arbitrary constant a; there will be altogether r equations, from which, if the fluxional coefficients be eliminated, there will result the primitive of the proposed equation involving the arbitrary constant a.

11. EXAMPLES.

Ex. 1. $y + ax + b = 0$.

There are two first derivatives, $\left.\begin{matrix} y - xp + b = 0 \\ p + a = 0 \end{matrix}\right\}$.

The second derivative is $q = 0$; or the equation admits not of a second derivative.

Ex. 2. $y + ax + c\sqrt{1 + a^2} = 0$.

The two first derivatives are $\left.\begin{matrix} y - xp + c\sqrt{1 + p^2} = 0 \\ p + a = 0 \end{matrix}\right\}$;

and the second is $q = 0$.

If p be eliminated from these, there results the original equation.

Ex. 3. $x^2 - 2ay - a^2 - b = 0$.

The *immediate* first derivative is $x - ap = 0$ (α); and eliminating a, which $= \frac{x}{p}$, another first derivative is

$$x^2 - \frac{2xy}{p} - \frac{x^2}{p^2} - b = 0, \text{ or } (x^2 - b)p^2 - 2xyp - x^2 = 0 \ (\beta),$$

from which may be deduced $p\sqrt{x^2 + y^2 - b} - yp - x = 0$ (γ).

If p be eliminated by combining (α) with either (β) or (γ), there will result the original equation.

The second derivative of this example may be deduced in four different ways.

First, differentiating (α), we have $1 - aq = 0$ or $a = \frac{1}{q}$, and, by substitution, $x - \frac{p}{q} = 0$, or $xq - p = 0$ is the second derivative.

Next, differentiate $a = \frac{x}{p}$, and there results

$$0 = \frac{1}{p} - \frac{xq}{p^2} \text{ or } xq - p = 0.$$

It may also be deduced by eliminating b from (γ).

For, differentiating,

$$q\sqrt{x^2 + y^2 - b} + \frac{p(x + yp)}{\sqrt{x^2 + y^2 - b}} - yq - p^2 - 1 = 0, \text{ in}$$

which, if we substitute for $\sqrt{x^2 + y^2 - b}$ its value $\frac{yp + x}{p}$,

there results $\frac{q(yp+x)}{p}+p^2-yq-p^2-1=0$ or $xq-p=0$.

Lastly, differentiating $x^2-\frac{2xy}{p}-\frac{x^2}{p^2}-b=0$, we have

$-\frac{y}{p}+\frac{xyq}{p^2}-\frac{x}{p^2}+\frac{x^2q}{p^3}=0$ or $\left\{\frac{xq}{p}-1\right\}\left\{\frac{y}{p}+\frac{x}{p^2}\right\}=0$;

and consequently $xq-p=0$ and $yp+x=0$; of which the latter is not a first derivative; it is the equation which results if we eliminate a and b by combining the primitive with (α) and (γ), taking the negative sign of the radical.

Ex. 4. $2xy+ay^2-bx^2=0$.

The two first derivatives are $\left.\begin{array}{l}(xp-y)(bx-\ y)=0\\(xp-y)(\ x+ay)=0\end{array}\right\}$;

which may be obtained either by differentiation and elimination; or by reducing the primitive to the forms

$\frac{2x}{y}+a-\frac{bx^2}{y^2}=0$, $\frac{2y}{x}+\frac{ay^2}{x^2}-b=0$; and then differentiating.

The equation being homogeneous does not admit of a second derivative; or $q=0$. (Art. 9.)

It appears from the first derivatives that $xp-y=0$ is also an equation of the first order derivable from the primitive. This equation may be deduced by the elimination of a and b; for, since $\frac{2x}{y}+a-\frac{bx^2}{y^2}=0$, therefore, differentiating $\frac{1}{y}-\frac{xp}{y^2}-\frac{bx}{y^2}+\frac{bx^2p}{y^3}=0 \therefore b=\frac{xpy-y^2}{x^2p-xy}$; and differentiating a second time, making $q=0$, we have

$0=\frac{-yp+xp^2}{x^2p-xy}-\frac{(xpy-y^2)(xp-y)}{(x^2p-xy)^2}=$ . . . . . .

$\frac{xp(xp-y)^2-y(xp-y)^2}{x^2(xp-y)^2}=\frac{xp-y}{x^2}$ or $xp-y=0$.

The same equation will result, if we eliminate b first and then a.

Ex. 5. $y^2=b(a^2-x^2)$.

The two first derivatives are $\left.\begin{array}{l}xy+p(a^2-x^2)=0\\yp+bx=0\end{array}\right\}$;

and the second derivative is $xyq+xp^2-yp=0$.

12. *The Integration of Fluxional Equations.*

Since the first derivatives contain one constant less than the primitive; the second derivatives, two less, and so on; it follows, è converso, that the canonical or complete primitive contains one arbitrary constant more than its derivative of the first order, two more than its derivative of the second order, and so on; and that it cannot contain a greater number of constants, because more could not have disappeared by differentiation and elimination.

If from a fluxional equation of the nth order, we can obtain by any process whatever an equation $u = 0$, which involves n arbitrary constants, $u = 0$ is called its *complete primitive* or *fluent*. It follows also, è converso, that every fluxional equation of the second order has two primitives of the first order, each involving a different arbitrary constant; and generally, those of the nth order have n different primitives of the $n - 1$th order. As these two propositions admit of direct demonstrations, they shall form the subject of the following articles.

13. *The complete value of y in terms of x as deduced from a fluxional equation of the nth order involves n arbitrary constants.*

Since $y = fx$, therefore (Maclaurin's Theor.)

$$y = f. + f.' \frac{x}{1} + f.'' \frac{x^2}{1.2} + f.''' \frac{x^3}{1.2.3} + \cdots \text{ of which}$$

$f.$ is a constant quantity; for it is the value of fx when $x = 0$.

First, let the equation be of the *first* order, $\mathrm{F}(x, y, p) = 0$.

Here p is a function of (x, y); and therefore $f.'$, which is the value of p when $x = 0$ and $y = f.$, is a function of $f.$

Differentiating $\mathrm{F}(x, y, p) = 0$, there results $f(x, y, p, q) = 0$; and supposing $x = 0$, we have $y = f.$ and p a function of $f.$; consequently (q) or $f.''$ is a function of $f.$

Similarly it may be shown that $f.'''$, $f.^{\mathrm{iv}}$, $\cdots$ are all functions of $f.$

Next let the equation be of the *second* order, $\mathrm{F}(x, y, p, q) = 0$.

Suppose $x = 0$, then it appears from the equation that $f.''$ is a function of $(f., f.')$; and by differentiating, it may be shown as before that $f.'''$, $f.^{\mathrm{iv}}$, $\cdots$ are all functions of $(f., f.')$; or the expression for y involves *two* constants; and so on.

This demonstration holds good even in those cases in which the developement given by Maclaurin's Theorem is only an analytical and not a numerical value of y; and the

reason is, that the number of the constants in $y = fx$ can neither be increased nor diminished by any analytical process. Vid. vol. i. ch. 2, 23.

Cor. It appears that the arbitrary constants possess this peculiar property, that when y is developed in a series by Maclaurin's Theorem, they are equal to $f.$, $f.'$, $f.''$, $\cdots$ respectively, the equations being continued as far as the number of the constants.

14. *To a fluxional equation of the nth order there belong n different primitives of the n − 1th order.*

For when $y = fx$ (Taylor's Theorem),

$$f(x + h) = y + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1.2} + r \cdot \frac{h^3}{1.2.3} + \cdots$$

Suppose $h = -x$; then $f(x + h) = f(x - x) = f.$, for $f(x - x)$ is the same as fx when $x = 0$; and the equation becomes

$$f. = y - p \cdot \frac{x}{1} + q \cdot \frac{x^2}{1.2} - r \cdot \frac{x^3}{1.2.3} + \cdots (\alpha).$$

For y substitute p, which is a function of x, and (α) becomes

$$f.' = p - q \cdot \frac{x}{1} + r \cdot \frac{x^2}{1.2} - \cdots (\beta).$$

Similarly in (α) substitute q, r, $\cdots$ for y, and there results

$$f.'' = q - r \cdot \frac{x}{1} + \cdots (\gamma).$$

$\cdots = \cdots$

First, let the proposed equation be of the first order, $F(x, y, p) = 0$.

By differentiation, it will appear as in the preceding article that p, q, r, $\cdots$ are functions of x, y; and if these values be substituted in (α), there will result an equation involving one arbitrary constant $f.$

Next, let the equation be $F(x, y, p, q) = 0$.

Here q, r, s, $\cdots$ are functions of x, y, p; and substituting their values in (β) and (γ), there will result two equations of the first order, which are primitives to the proposed equation; and so on.

15. If then from an equation of the second order there can be deduced by any process whatever two equations of the first order, each involving an arbitrary constant, these are the first primitives of the proposed equation; and every other equation of the first order from which the proposed

equation can be derived must be a particular case of one of these primitives.

When these primitives are determined, eliminating p, there will result an equation which involves the two constants, and which is the canonical fluent, or original primitive of the proposed equation in as general a form as possible.

Generally, if from an equation of the nth order, there can be deduced n equations of the $n - 1$th order, each involving an arbitrary constant; eliminating all the fluxional coefficients, there will result the original fluent involving n constants.

CHAPTER IV.

Fluxional Equations of two Variables of the first order and degree.

1. THE two principal methods of integrating a fluxional equation which contains two or more variables are (1), by separating the variables, and then integrating each term; and (2), by finding a factor which renders the equation an exact fluxion.

When the variables can be separated, the equation is said to be integrable, as it is supposed that the fluents can be obtained by the common rules.

It may be here observed, in general, that it is usual with analysts to consider any problem as solved which has been reduced to one of a lower class. Thus, an equation of the second order is said to be integrated, if the problem can be made to depend upon one of the first order, though this latter equation may be such that we have not the means of integrating it.

2. When the fluents are transcendentals; the algebraick relation between the variables may be sometimes found.

Ex. 1. $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0.$

Integrating, and correcting the fluents by a like quantity, we have $\sin.^{-1}x + \sin.^{-1}y = \sin.^{-1}c$; and to find the algebraick relation between x and y, let $\sin.^{-1}x = \mathrm{X}$, $\sin.^{-1}y = \mathrm{Y}$; then $c = \sin.(\mathrm{X} + \mathrm{Y}) = \sin.\mathrm{X}\cos.\mathrm{Y} + \cos.\mathrm{X}\sin.\mathrm{Y}$ . . . $= x\sqrt{1-y^2} + y\sqrt{1-x^2}$.

Ex. 2. $d\tan.^{-1}x + d\tan.^{-1}y + d\tan.^{-1}z = 0.$

Integrating, $\tan.^{-1}c = \tan.^{-1}x + \tan.^{-1}y + \tan.^{-1}z$; or

$$c = \frac{x+y+z-xyz}{1-(xy+xz+yz)}.$$ (Trig. p. 36.)

3. SEPARATION OF THE VARIABLES.

All equations of the form $\mathrm{X}dy + \mathrm{Y}dx = 0$, *where* X *and* Y *are functions of* x *and* y, *are integrable.*

For, dividing by xy, $\frac{dy}{y} + \frac{dx}{x} = 0$; in which the variables are separated.

Ex. 1. $ydx - xdy = 0$.

Dividing by xy, $\frac{dx}{x} - \frac{dy}{y} = 0$; and integrating, $lx - ly = lc$ or $x = cy$.

Ex. 2. $(1 + x^2)dy = y^{\frac{1}{2}}dx$.

Here $\frac{dy}{y^{\frac{1}{2}}} = \frac{dx}{1+x^2}$ and $2y^{\frac{1}{2}} = \tan.^{-1}x + c$.

Ex. 3. $\sqrt{b^2 - y^2}dx = \sqrt{2ax - x^2}dy$.

Here $\frac{dy}{\sqrt{b^2-y^2}} = \frac{dx}{\sqrt{2ax-x^2}}$ $\therefore$ $\sin.^{-1}\frac{y}{b} = v.s.^{-1}\frac{x}{a}$ . . . $= \sin.^{-1}\sqrt{\frac{2x}{a} - \frac{x^2}{a^2}}$ $\therefore y = \frac{b}{a}\sqrt{2ax - x^2}$.

This solution is not the complete fluent of the proposed equation, since it does not contain an arbitrary constant.

4. *All equations of the form* $xydy + x'y'dx = 0$ *where* x' *and* y' *are any functions of* x *and* y, *are integrable.*

For, dividing by xy', $\frac{ydy}{y'} + \frac{x'dx}{x} = 0$; where the variables are separated.

Ex. 1. $x^2ydx + (3y + 1)x^{\frac{3}{2}}dy = 0$.

Dividing by $x^{\frac{3}{2}}y$, $x^{\frac{1}{2}}dx = -\frac{3y+1}{y}dy$ $\therefore$ &c.

5. *An equation is integrable, if, by any artifice of calculation, one side of the equation may be rendered an exact fluxion, and the other a function which contains only one of the variables.*

Ex. 1. $\frac{xdx+ydy}{x^2+y^2} = ydy$.

Multiplying by 2, $\frac{2xdx+2ydy}{x^2+y^2} = 2ydy$ $\therefore$ . . . . . $l(x^2 + y^2) = 2\int ydy + c$.

Ex. 2. $\frac{xdy+ydx}{x} = -dy.$

Dividing by y, $\frac{xdy+ydx}{xy} = -\frac{dy}{y}$ $\therefore l.xy = l\frac{c^2}{y}$ or $xy^2 = c^3$.

Ex. 3. $adx = ydy - xdy.$

Subtracting ady, $a(dx - dy) + (x - y + a)dy = 0$ $\therefore$ $dy = -\frac{a(dx-dy)}{x-y+a}$ $\therefore \frac{y}{a} = l\frac{c}{x-y+a}$ or $x = ce^{-\frac{y}{a}} + y - a.$

Ex. 4. $adx^2 = ydy^2 - bdxdy.$

Here the variables may be separated by the solution of a quadratick; and we have $dx^2 + \frac{b}{a}dydx + \frac{b^2dy^2}{4a^2}$ . . . $= \frac{4ay+b^2}{4a^2}dy^2$ $\therefore dx + \frac{bdy}{2a} = \frac{\sqrt{4ay+b^2}}{2a}dy$ $\therefore x + \frac{by}{2a}$ . . . $= \frac{(4ay+b^2)^{\frac{3}{2}}}{12a^2} + c.$

Ex. 5. $ydx = y^3dy + y^2dy + xdy.$

Here $\frac{ydx-xdy}{y^2} = ydy + dy$ $\therefore \frac{x}{y} = \frac{y^2}{2} + y.$

Ex. 6. $a^2dx = (x+y)^2dy.$

$\frac{dx}{dy} = \frac{(x+y)^2}{a^2}$ $\therefore \frac{dx+dy}{dy} = \frac{(x+y)^2+a^2}{a^2}$ $\therefore \frac{dx+dy}{(x+y)^2+a^2}$ . . . $= \frac{dy}{a^2}$ $\therefore$ integrating, $\tan.^{-1}\frac{(x+y)}{a} = \frac{y+c}{a}$ or . . . . . $x + y = a\tan.\frac{y+c}{a}.$

This example may be integrated by substituting $z = x + y$.

Ex. 7. $(xdy + ydx)\sqrt{a^4 - x^2y^2} = \frac{xdx+ydy}{(x^2+y^2)^{\frac{3}{4}}}.$

The first member of the equation is of the form $dz\sqrt{a^2-z^2}$ which is a circular area; and the fluent of the second is $2(x^2 + y^2)^{\frac{1}{4}}$.

Ex. 8. $dy - (x + y)^n dx = a^n dx.$

Substitute $z = x + y$ $\therefore dz - dx - z^n dx = a^n dx$ $\therefore$ . . .

$dx = \dfrac{dz}{z^n + a^n + 1} \therefore x = \int \dfrac{dz}{z^n + a^n + 1}$.

Ex. 9. $dz = x^2dx + xydy + y^2dx$ where $z = \text{F}x$.

Here $xdz = x^3dx + x^2ydy + xy^2dx$ or $xdx(x^2 + y^2) + x^2ydy = xdz \therefore (xdx + ydy)(x^2 + y^2) = xdz + y^3dy \therefore \frac{1}{4}(x^2 + y^2)^2 = \frac{1}{4}y^4 + \int xdz$.

Ex. 10. $xdy - ydx = \dfrac{2xdy - ydx}{\sqrt{x^2 + y^2}}$.

Substitute $x = yz \therefore yzdy - y(zdy + ydz)$ . . . .

$= \dfrac{2yzdy - y(zdy + ydz)}{y\sqrt{1 + z^2}}$ or $-y^2dz = \dfrac{zdy - ydz}{\sqrt{1 + z^2}} \therefore$ . . .

$dz\sqrt{1 + z^2} = \dfrac{ydz - zdy}{y^2} \therefore \dfrac{z}{y} = \int dz\sqrt{1 + z^2}$ . . . . .

$= \frac{1}{2}z\sqrt{1 + z^2} + \frac{1}{2}l(z + \sqrt{1 + z^2}) + c$ (vol. i. ch. 2, 55, ex. 1), or $\dfrac{x}{y^2} = \dfrac{x}{2y^2}\sqrt{x^2 + y^2} + \frac{1}{2}l\dfrac{x + \sqrt{x^2 + y^2}}{y} + c \therefore$ . . . .

$2x = x\sqrt{x^2 + y^2} + y^2l\dfrac{x + \sqrt{x^2 + y^2}}{y} + cy^2$.

Ex. 11. $ydy - xdx = a(xdy + ydx)$.

Multiplying by 2, we have $d.(y^2 - x^2) = 2ad.xy$; and integrating $y^2 - x^2 = 2axy + c$.

Ex. 12. $y^2dx - xydy = \text{Y}\sqrt{y^2dx^2 - 2xydxdy + y^2dy^2}$.

Substitute $x = zy \therefore dz = \dfrac{ydx - xdy}{y^2}$ and

$y^3dz = y^2dx - xydy$.

Also $dx^2 = z^2dy^2 + 2zydydz + y^2dz^2$

$\therefore y^2dx^2 = z^2y^2dy^2 + 2zy^3dydz + y^4dz^2$

and $2xydxdy = (2zy^2dydx) = 2z^2y^2dy^2 + 2zy^3dydz$.

$\therefore y^2dx^2 - 2xydxdy = y^4dz^2 - z^2y^2dy^2$; whence by substitution the equation becomes $y^3dz = \text{Y}\sqrt{y^4dz^2 - z^2y^2dy^2 + y^2dy^2}$; or dividing by y and squaring both sides, $y^4dz^2 = \text{Y}^2y^2dz^2 - \text{Y}^2(z^2 - 1)dy^2 \therefore \text{Y}^2(z^2 - 1)dy^2 = y^2(\text{Y}^2 - y^2)dz^2$; and as in art. 4, $\dfrac{dz^2}{z^2 - 1} = \dfrac{\text{Y}^2dy^2}{y^2(\text{Y}^2 - y^2)}$; where the variables are separated.

6. *If u is an homogeneous algebraical function of (x, y) which rises to n dimensions* $u = x^n \cdot \varphi\left(\frac{y}{x}\right)$.

For if u contain a multinomial term of the form $(ex^ry^s + fx^{r'}y^{s'} + \text{&c.})^k$, it may be expanded by the binomial theorem into simple terms of the form $ex^{kr}y^{ks} + \text{&c.}$ where $k(r+s) = n$; hence we may assume $u = ax^py^q + bx^{p'}y^{q'} + cx^{p''}y^{q''} + \cdots$ where $n = p+q = p'+q' = p''+q'' = \text{&c.}$; wherefore, dividing by x^n, we have

$$\frac{u}{x^n} = \frac{ay^q}{x^{n-p}} + \frac{by^{q'}}{x^{n-p'}} + \frac{cy^{q''}}{x^{n-p''}} + \cdots$$

$$= a\left(\frac{y}{x}\right)^q + b\left(\frac{y}{x}\right)^{q'} + c\left(\frac{y}{x}\right)^{q''} + \cdots;$$

or $u = x^n \cdot \varphi\left(\frac{y}{x}\right)$.

Generally, if u is an homogeneous function of any number of variables x, y, z, $\cdots$ of n dimensions, it may be shown that $u = x^n \cdot \varphi\left(\frac{y}{x}, \frac{z}{x}, \cdots\right)$.

For any term as $x^py^qz^r$, where $p+q+r = n$, being divided by x^n gives $\frac{y^qz^r}{x^{q+r}} = \left(\frac{y}{x}\right)^q \cdot \left(\frac{z}{x}\right)^r$ which $\cdots$

$= \varphi\left(\frac{y}{x}, \frac{z}{x}\right)$.

Cor. 1. By assuming $t = \frac{y}{x}$, $v = \frac{z}{x}$, $\cdots$ an homogeneous function may be transformed into one which shall contain one variable less than the original.

Cor. 2. The theorem of the article does not necessarily include transcendentals; for logarithmick and circular functions when expanded cease to be homogeneous.

7. *If* $du = \text{A}dx + \text{B}dy + \text{C}dz + \cdots$, *where* A, B, C, $\cdots$ *are homogeneous functions of* $(x, y, z, \cdots)$ *of* $n-1$ *dimensions; then shall* $\text{A}x + \text{B}y + \text{C}z + \cdots = nu$.

For, in the function u suppose that there be substituted $y = tx$, $z = sx$, &c. = &c.; then it takes the form $u = \text{P}x^n$ where $\text{P} = \text{F}(t, s, \cdots)$.

Let $d\text{P} = pdt + qds + \cdots$; then du, which $= x^n d\text{P} + n\text{P}x^{n-1}dx = n\text{P}x^{n-1}dx + x^n(pdt + qds + \cdots)$.

But $du = \text{A}dx + \text{B}(tdx + xdt) + \text{C}(sdx + xds) + \cdots$; wherefore, $n\text{P}x^{n-1}dx + px^n dt + qx^n ds + \cdots = (\text{A} + \text{B}t + \text{C}s + \cdots)dx + \text{B}xdt + \text{C}xds + \cdots$; and, equating coefficients, $n\text{P}x^{n-1} = \text{A} + \text{B}t + \text{C}s + \cdots = \text{A} + \frac{\text{B}y}{x} + \frac{\text{C}z}{x} + \ldots$

or $\text{A}x + \text{B}y + \text{C}z + \cdots = n\text{P}x^n = nu$.

Cor. 1. $nu = \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z + \cdots$

Cor. 2. When $n = 0$, $\frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z + \cdots = 0$.

Cor. 3. $(n-1)du = x d\text{A} + y d\text{B} + z d\text{C} + \cdots$

For, since $nu = \text{A}x + \text{B}y + \cdots$, we have $ndu = \text{A}dx + \text{B}dy + \cdots + x d\text{A} + y d\text{B} + \cdots = du + x d\text{A} + y d\text{B} + \cdots$ or $(n-1)du = x d\text{A} + y d\text{B} + \cdots$

8. *The proposition contained in cor.* 2 *of the preceding article may be proved è converso.*

For, let $u = f(x, y)$ be the function which belongs to the proposed equation; then, since $\frac{\dot{du}}{dx} = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}$, we have by substitution, $\frac{\dot{du}}{dx} = \frac{du}{dy}\left\{\frac{dy}{dx} - \frac{y}{x}\right\}$ . . . . . or $du = \frac{du}{dy}x \times \frac{xdy - ydx}{x^2} = \frac{du}{dy}x \times d\left(\frac{y}{x}\right)$; which being an exact fluxion, it follows that $\frac{du}{dy}x = \phi\left(\frac{y}{x}\right)$ * and . . . $du = \varphi\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$; and consequently $u = \text{F}\left(\frac{y}{x}\right)$.

9. *Required to investigate the relation between an homogeneous function of given dimensions, and its fluxional coefficients of any order.*

Let $u =$ an homogeneous function of $(x, y, z, \cdots)$ of n

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* This is the converse of the proposition established in vol. i. ch. 1.; viz. if $u = \text{F}x$, $du = \phi x dx$. It follows immediately from the nature of fluxions.

dimensions; for $x, y, \cdots$, substitute $tx, ty, \cdots$; then, since the function is homogeneous and of n dimensions, u becomes $t^n u$, and the equation becomes $t^n u = \mathrm{F}(tx, ty, \cdots)$.

In this equation, suppose t to become $t + h$, t at the same time being made $= 1$; then (vol. i. ch. 4, 20).

$$(1+h)^n u = u + \frac{1}{1}\left\{\frac{du}{dx}hx + \frac{du}{dy}hy + \frac{du}{dz}hz\right\}$$
$$+\frac{1}{1.2}\left\{\frac{d^2u}{dx^2}h^2x^2+\frac{d^2u}{dy^2}h^2y^2+\frac{d^2u}{dz^2}h^2z^2+\frac{d^2u}{dxdy}2h^2xy\right.$$
$$\left.+\ \&c.;\qquad\qquad +\ \&c.\right\}$$

whence, expanding $(1 + h)^n u$ and equating terms which include the same powers of the indeterminate quantity h; we have

$$nu = \frac{du}{dx}x + \frac{du}{dy}y + \frac{du}{dz}z + \cdots$$
$$n(n-1)u = \frac{d^2u}{dx^2}x^2 + \frac{d^2u}{dy^2}y^2 + \frac{d^2u}{dz^2}z^2 + \frac{2d^2u}{dxdy}xy + \frac{2d^2u}{dxdz}xz \cdots$$
$$\&c. = \&c. \qquad\qquad + \frac{2d^2u}{dydz}yz.$$

10. *Homogeneous equations of the first order and degree are always integrable.*

For, let $\mathrm{P}dx + \mathrm{Q}dy = 0$ be an homogeneous equation in which P and Q are of n dimensions; in it substitute $\frac{y}{x} = z$ or $y = zx$; then (art. 6) $\mathrm{P} = x^n \mathrm{F}z$ and $\mathrm{Q} = x^n f z$; wherefore by substitution, the equation becomes, when divided by x^n, $\mathrm{F}z.dx + fz.dy = 0$; and to eliminate dy, we have $dy = zdx + xdz$, and consequently $\mathrm{F}zdx + fz\{zdx + xdz\} = 0$; or $\frac{dx}{x} + \frac{fz.dz}{\mathrm{F}z + zfz} = 0$; where the variables are separated, and the integration can be effected.

Ex. 1. $y^2dx + xydx - x^2dy = 0$.

Substituting $y = xz$ and dividing by x^2, ...

$z^2dx + zdx - (zdx + xdz) = 0 \therefore \frac{dx}{x} = \frac{dz}{z^2}$ and $c = xe^{\frac{x}{y}}$.

Ex. 2. $xdx + ydy = nydx$.

Substitute $y = xz$, and dividing by x, ...

$dx + z(xdz + zdx) = nzdx$ or $\frac{dx}{x} = \frac{-zdz}{1-nz+z^2}$ . . . .

$= \frac{\frac{1}{2}ndz - zdz}{1-nz+z^2} - \frac{\frac{1}{2}ndz}{1-nz+z^2}$; and, integrating,

$lx = -\frac{1}{2}l(1 - nz + z^2) - \frac{n}{2}\int \frac{dz}{1-nz+z^2}$.

Here n may be either $>$, $=$ or $<$ 2.

1. $n > 2$.

Let a and $\frac{1}{a}$ be the roots of $z^2 - nz + 1 = 0$ (Alg. 292);

then $n = a + \frac{1}{a} = \frac{a^2+1}{a}$; and $\frac{1}{1-nz+z^2}$ . . . . .

$= \frac{a}{a^2-1}\left\{\frac{1}{z-a} - \frac{1}{z-\frac{1}{a}}\right\}$ $\therefore lx = -\frac{1}{2}l(1 - nz + z^2)$

$-\frac{a^2+1}{2(a^2-1)}\, l\, \frac{z-a}{z-\frac{1}{a}} = -\frac{1}{2}l(1-nz+z^2) - \frac{a^2+1}{2(a^2-1)} l \frac{az-a^2}{az-1}$

$= -\frac{1}{2}l\left(1 - \frac{ny}{x} + \frac{y^2}{x^2}\right) - \frac{a^2+1}{2(a^2-1)}\, l\, \frac{ay-a^2x}{ay-x}$ $\therefore$ . . .

$l\sqrt{x^2 - nxy + y^2} + \frac{a^2+1}{2(a^2-1)}\, l\, \frac{ay-a^2x}{ay-x} = c.$

2. $n = 2$.

Here $\int \frac{dz}{1-nz+z^2} = \int \frac{dz}{(1-z)^2} = \frac{1}{1-z}$ $\therefore lx(1 - z)$ . . .

$= c - \frac{1}{1-z}$ or $l(x - y) + \frac{x}{x-y} = c.$

3. $n < 2$.

Let $n = 2\cos.\theta$; then $1 - nz + z^2 = (z - \cos.\theta)^2 + \sin.^2\theta$

$\therefore \int \frac{dz}{1-nz+z^2} = \frac{1}{\sin.\theta}\tan.^{-1}\frac{z-\cos.\theta}{\sin.\theta} =$, adding

$\frac{1}{\sin.\theta}\tan.^{-1}\frac{\cos.\theta}{\sin.\theta}$ in order to correct from $z = 0$, as in ch. 1, 8,

$\frac{1}{\sin.\theta}\tan.^{-1}\frac{z\sin.\theta}{1-z\cos.\theta}$ $\therefore l\sqrt{x^2-nxy+y^2}$ . . . . . .

$+ \frac{\cos.\theta}{\sin.\theta}\tan.^{-1}\frac{y\sin.\theta}{x-y\cos.\theta} = c.$

Ex. 3. $x^2dy = y^2dx + x^2dx$.

Substituting $y = xz$, and dividing by x^2, $zdx + xdz \cdots$

$= z^2dx + dx \therefore xdz = (z^2 - z + 1)dx \therefore \frac{dx}{x} = \frac{dz}{z^2 - z + 1} \cdots$

$\therefore lx = \frac{2}{\sqrt{3}} \tan.^{-1} \frac{z\sqrt{3}}{2 - z} \therefore$ &c.

Ex. 4. $y^2dy = 3yxdx - x^2dy$.

Here it will be more convenient to substitute $x = yz$; and dividing by y^2, $dy = 3z(ydz + zdy) - z^2dy = 3yzdz + 2z^2dy$

$\therefore \frac{dy}{y} = \frac{3zdz}{1 - 2z^2}$ and $ly = \frac{3}{4} l \frac{c}{1 - 2z^2} \therefore y^4 = \frac{c^3}{(1 - 2z^2)^3} \cdots$

$= \frac{c^3y^6}{(y^2 - 2x^2)^3}$; or $y^2 - 2x^2 = cy^{\frac{2}{3}}$ is the required equation.

Ex. 5. $aydx = (ax - \sqrt{x^2 + y^2})dy$.

Substitute $x = yz$; and dividing by y, $a(ydz + zdy) \cdots$

$= (az - \sqrt{z^2 + 1})dy$ or $-\frac{dy}{y} = \frac{adz}{\sqrt{1 + z^2}} \therefore \cdots$

$\frac{c}{y} = (z + \sqrt{1 + z^2})^a = \frac{(x + \sqrt{x^2 + y^2})^a}{y^a} \therefore \cdots$

$cy^{a-1} = (x + \sqrt{x^2 + y^2})^a$.

Ex. 6. $xydy - y^2dx = (x + y)^2 e^{-\frac{y}{x}} dx$.

Substituting $y = xz$ and dividing by x^2, $z(zdx + xdz) - z^2dx = (1 + z)^2 e^{-z} dx \therefore \frac{dx}{x} = \frac{e^z zdz}{(1 + z)^2} \therefore$ (ch. i. art. 42, cor.)

$l \frac{x}{c} = \frac{e^z}{1 + z} = \frac{e^{\frac{y}{x}}}{1 + \frac{y}{x}}$ or $(x + y) l \frac{x}{c} = xe^{\frac{y}{x}}$.

Ex. 7. $\frac{dx}{x} + \frac{y^2dx}{x^3} - \frac{ydy}{x^2} + \frac{ydx - xdy}{x^3} \sqrt{x^2 + y^2} + \frac{dy}{2y} = 0$.

Substituting $y = xz$ and multiplying by x,

$dx + z^2dx - z(xdz + zdx) + \{zdx - (xdz + zdx)\} \sqrt{1 + z^2}$

$+ \frac{dx}{2} + \frac{xdz}{2z} = 0$ or $\frac{3dx}{2} - zxdz - xdz \sqrt{1 + z^2} + \frac{xdz}{2z} = 0$

or $\frac{3dx}{2x} = \left(z + \sqrt{1 + z^2} + \frac{1}{2z}\right) dz \therefore$ &c.

Ex. 8. $(x^2 - ay^2)dx - x^2dy = 0$.

Here $\left(1 - \frac{ay^2}{x^2}\right)dx - dy = 0$; substitute $\frac{y}{x} = z$ or $y = zx \therefore (1 - az^2)dx = zdx + xdz$ or $\frac{dx}{x} = \frac{dz}{1-z-az^2}$ $\therefore$ &c.

11. Examples of fluxional equations which are not homogeneous, but which may be rendered such by proper substitutions.

Ex. 1. Let $(ax + by + c)dx + (ex + fy + g)dy = 0$, which is a canonical form.

Assume $ax + by + c = v$, and $ex + fy + g = w$ $\therefore$

$\left.\begin{array}{l} adx + bdy = dv \\ edx + fdy = dw \end{array}\right\}$ $\therefore$ by elimination, $dx = \frac{fdv - bdw}{af - be}$ and $dy = \frac{adw - edv}{af - be}$, $\therefore$ by substitution and multiplying by $af - be$, the proposed equation becomes $v(fdv - bdw) + w(adw - edv) = 0$ or $(fv - ew)dv + (aw - bv)dw = 0$; which is homogeneous.

Otherwise. Assume $x = v + m$ and $y = w + n$; then the equation becomes $(av + bw + am + bn + c)dv + (ev + fw + em + fn + g)dw = 0$; and if we suppose $am + bn + c = 0$ and $em + fn + g = 0$, from which the requisite values of m and n may be obtained, there will result an equation which is homogeneous.

Cor. 1. If $b = e$, this equation is immediately integrable without substitution, as will be shown in art. 22.

Cor. 2. If $af = be$, then $m = \infty$ and $n = \infty$, and the example fails; but in this case the proposed equation may be reduced to the form $(ax + by)dx + (ax + by)dy + cdx + gdy = 0$; in which substitute $v = ax + by$ and the resulting value of dy, and there will arise an equation in which the variables may be separated.

Ex. 2. $axdx + 2aydx + bxdy - abdy = 0$.

This equation being of the first order and degree with respect to x and y, is integrable by either of the methods of the preceding example.

Assume $x = v + m$ and $y = w + n$, $\therefore$ $(av + 2aw + am + 2an)dv + (bv + bm - ab)dw = 0$, $\therefore$ we have

$\left.\begin{array}{l} am + 2an = 0 \\ bm - ab = 0 \end{array}\right\}$ $\therefore$ $m = a$, $n = -\frac{a}{2}$; and the pro-

posed equation becomes $avdv + 2awdv + bvdw = 0$ where $v = x - a$ and $w = y + \frac{a}{2}$.

Ex. 3. $2axdx - 2bydy - 4aydx + bxdy - a^2dx = 0$ or $(2ax - 4ay - a^2)dx + (bx - 2by)dy = 0$.

If this be compared with the canonical form, it appears that $af = be$; but if we assume $v = 2ax - 4ay$ (ex. 1, cor. 2), $y = \frac{2ax - v}{4a}$ and $dy = \frac{2adx - dv}{4a}$; and by substitution, there results $(v - a^2)dx + \frac{bv}{2a} \cdot \frac{2adx - dv}{4a}$; and by reduction,

$$dx = \frac{bvdv}{8a^2v + 2abv - 8a^4}.$$

Ex. 4. $\sqrt{a^2x^2 + ay^3} \cdot dx = y^2dy$.

Substitute $y^3 = av^2$, and the equation is rendered homogeneous.

Ex. 5. $x^3dx + \frac{x^2dy}{\sqrt{a+y}} = dy$.

Substitute $\sqrt{a + y} = w$, $\therefore x^3dx + 2x^2dw = 2wdw$. Again, substitute $x^2 = v$, $\therefore \frac{1}{2}vdv + 2vdw = 2wdw$.

Ex. 6. $adx + (bx + cy)dx = edy + (bx + cy)ndx$ or $adx - s(bx + cy)dx = edy$, if $s = n - 1$.

Substitute $v = bx + cy$, and the equation becomes

$$(a - sv)dx = \frac{e}{c}(dv - bdx) \therefore (ac + be - scv)dx = edv$$

$$\therefore dx = \frac{edv}{ac + be - scv} \therefore \text{\&c.}$$

Ex. 7. $\frac{dy}{dx} = \frac{a + 2x - y}{a - x + 2y}$.

By substituting $x = v - a$ and $y = w - a$, the equation becomes $(2v - w)dv + (v - 2w)dw = 0$, which being homogeneous is integrable; and there results $c^4 = (x - y)(2a + x + y)^3$.

Ex. 8. $e^xdx + \frac{ydy}{e^x} = dy - ydx$.

Substitute $v = e^x \therefore dv + \frac{ydy}{v} = dy - \frac{ydv}{v}$, which is homogeneous; and the primitive is $\tan.^{-1} y.e^{-x} = c + l\sqrt{e^{2x} + y^2}$.

Ex. 9. $e^x dx - \frac{ydy}{e^x} = dy - ydx$ may be integrated by the same substitution.

Or thus, $(e^x + y)dx = \left(1 + \frac{y}{e^x}\right)dy \therefore e^x dx = dy \therefore \cdots$ $e^x = y + c$. Here $e^x + y = 0$ is an equation which satisfies the proposed, but it is not one of its fluents.

12. Equations may be sometimes integrated by substituting $x = \text{A}y + v + \text{B}$; and by assuming A and B properly, coefficients of the resulting equation may be made to vanish, which will render the remainder integrable.

When the equation is of high dimensions, this process becomes very laborious. To illustrate the principle, we shall take Ex. 1 of the last article.

Ex. 1. $(ax + by + c)dx + (ex + fy + g)dy = 0$.

Let $x = \text{A}y + v + \text{B}$; then, by substitution,

$\{a(\text{A}y + v + \text{B}) + by + c\}(\text{A}dy + dv) + \{e(\text{A}y + v + \text{B}) + fy + g\}dy = 0$; or

$$\left.\begin{matrix}\text{A}(\text{A}a + b)\\ +\text{A}e + f\end{matrix}\right\}ydy + \text{A}a\left\{\begin{matrix}\\ +e\end{matrix}\right\}vdy + (\text{A}a + b)ydv + avdv \cdots$$
$$\left.\begin{matrix}+\text{AB}a + \text{A}c\\ +\text{B}e + g\end{matrix}\right\}dy + (\text{B}a + c)dv = 0.$$

If the 2d and 3d terms could be destroyed, the equation would be immediately integrable; but the equations of condition are $\text{A}a + e = 0$ and $\text{A}a + b = 0$; or $b = e$; in which case, as will be shown (art. 22), the proposed is integrable without substitution.

If the two last terms be destroyed, the equation becomes homogeneous, and consequently integrable; but the equations of condition are $\text{A}(\text{B}a + c) + \text{B}e + g = 0$ and $\text{B}a + c = 0$; wherefore $\text{B}e + g = 0$; and eliminating B, $ag = ce$, so that the two last terms cannot be made to vanish unless this condition obtains.

But if we destroy the 1st and 5th terms, the proposed becomes $(\text{A}a + e)vdy + (\text{A}a + b)ydv + avdv + (\text{B}a + c)dv = 0$; which is a *linear* equation, and is integrable by the formula of art. 15; and the equations of condition are $\text{A}^2a + \text{A}(b + e) + f = 0$, and $\text{B}(\text{A}a + e) + \text{A}c + g = 0$, which are two independent equations by means of which A and B may be determined.

Ex. 2. $abx^2dx + b^2yxdx + a^3ydx + a^2bydy + a^3xdy = 0$.

Let $y = \text{A}x + v + \text{B}$; then, by substitution,

$\{abx^2 + (b^2x + a^3)(\text{A}x + v + \text{B})\}dx + \cdots$

$\{a^2b(\text{A}x+v+\text{B})+a^3x\}(\text{A}dx+dv)=0$; or $(ab+\text{A}b^2)x^2dx+b^2vxdx+(\text{B}b^2+2\text{A}a^3+\text{A}^2a^2b)xdx+(a^3+\text{A}a^2b)(\text{B}dx+vdx)+(\text{A}a^2b+a^3)xdv+a^2bvdv+\text{B}a^2bdv=0$.

First, suppose $a+\text{A}b=0$, or $\text{A}=-\frac{a}{b}$; which will cause the first, fourth, and fifth terms to vanish.

Next, suppose $\text{B}b^2+2\text{A}a^3+\text{A}^2a^2b=0$; or eliminating A, $\text{B}b^2-\frac{a^4}{b}=0$ or $\text{B}=\frac{a^4}{b^3}$; then the proposed becomes

$b^2vxdx+a^2bvdv+\frac{a^6dv}{b^2}=0$ or $b^4xdx+a^2b^3dv+\frac{a^6dv}{v}=0$, which is immediately integrable.

13. PRAXIS.

1. $(6xy-y^2)dx+(3x^2-2xy)dy=0 \therefore 3x^2y-y^2x=c^3$.
2. $xdx+aydx+ydy=0 \therefore$ . . . . . $l\sqrt{x^2+axy+y^2}=\frac{a}{2}\int\frac{dz}{1+az+z^2}$.
3. $y^2dx-xydx+x^2dy=0 \therefore x=ce^{\frac{x}{y}}$.
4. $y^2dx+xydy+x^2dy=0 \therefore y^2x=c^2(x+2y)$.
5. $ydy+(x+2y)dx=0 \therefore l\frac{x+y}{c}+\frac{x}{x+y}=0$.
6. $x^2ydx-y^3dy=x^3dy \therefore x^3=3y^3l\frac{y}{c}$.
7. $dy\sqrt{x^2+y^2}=ydx \therefore \frac{dy}{y}=\sqrt{1+z^2}dz+zdz \therefore$ &c.
8. $(ax+y)dx=(x-ay)dy \therefore \text{tan.}^{-1}\frac{y}{x}=al\frac{\sqrt{x^2+y^2}}{c}$.
9. $xdy-ydx=dx\sqrt{x^2-y^2} \therefore l\frac{x}{c}=\text{sin.}^{-1}\frac{y}{x}$.
10. $xdy-ydx=dx\sqrt{x^2+y^2} \therefore x^2=c^2+2cy$.
11. $\frac{dx}{x}-\frac{dy}{y}:\frac{dx}{y}-\frac{dy}{x}::n:1 \therefore (y-x)^{n-1}(y+x)^{n+1}=c^{2n}$.
12. $x^{\frac{1}{2}}dy=y^{\frac{1}{2}}(dx+dy) \therefore$ . . . . . .

$$l\frac{c}{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y} = \tfrac{4}{3}\tan.^{-1}\left\{\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}-\frac{1}{2}\right\}.$$

13. $xdy-ydx = yl\frac{y}{x}dx \therefore y = xe^{\frac{x}{c}}$.

14. Fluxional equations of the first order and degree may be classed according to the number of terms which they contain. If the equation contain only two, the variables are always separable; thus, let $ax^my^ndy = bx^py^qdx$; dividing by x^my^q, we have $ay^{n-q}dy = bx^{p-m}dx$.

If the equation contain three terms, it may be reduced to the form $dy + by^ndx = ax^mdx$.

For let the equation be $cx^ky^ldy + bx^py^qdx = ax^ry^sdx$; then dividing by cx^ky^s, we have

$y^{l-s}dy + \frac{b}{c}y^{q-s}x^{p-k}dx = \frac{a}{c}x^{r-k}dx$. In this equation substitute $y^{l-s}dy = \frac{dw}{l-s+1}$ and $x^{p-k}dx = \frac{dv}{p-k+1}$; and there results $dw + \frac{b(l-s+1)}{c(p-k+1)}w^{\frac{q-s}{l-s+1}}dv = \frac{a(l-s+1)}{c(p-k+1)}v^{\frac{r-k}{p-k+1}}dv$; which is of the proposed form.

If $n = 1$, the equation is linear with respect to y.

If $n = 2$, the form is $dy + by^2dx = ax^mdx$; which is known by the name of Riccati's equation, who was the first that separated the variables in it.

15. *The linear equation* $dy + \text{P}ydx = \text{Q}dx$, *where* P *and* Q *are functions of* x, *is integrable.*

For, assume $y = \text{x}z$; then $dy = \text{x}dz + z\text{d}x$; and by substitution, $\text{x}dz + z\text{d}x + \text{P}\text{x}zdx = \text{Q}dx$ or $\text{x}(dz + \text{P}zdx) + z\text{d}x - \text{Q}dx = 0$.

But since x is undetermined, it may be assumed such that $dz + \text{P}zdx = 0$, and consequently $z\text{d}x - \text{Q}dx = 0$.

From the first, $\frac{dz}{z} = -\text{P}dx$; and integrating and correcting, $l\frac{z}{c} = -\int\text{P}dx$ or $z = ce^{-\int\text{P}dx}$.

From the second, $d\text{x} = \frac{\text{Q}dx}{z} = \frac{1}{c}\text{Q}e^{\int\text{P}dx}dx$; wherefore in-

tegrating, $x = \frac{1}{c}\int Qe^{\int Pdx}dx + c'$; and substituting these values of x and z in the original assumption $y = xz$, we have $y = e^{-\int Pdx}\left\{\int Qe^{\int Pdx}dx + c\right\}$.

Cor. 1. Equations are integrable of the form
$dy + Pydx = Qy^{n+1}dx$.

For substitute $z = \frac{1}{y^n}$ or $y = z^{-\frac{1}{n}}$; then

$-\frac{1}{n}z^{-\frac{1}{n}-1}dz + Pz^{-\frac{1}{n}}dx = Qz^{-\frac{1}{n}-1}dx$ or . . . . . .

$dz - nPzdx = -nQdx$; and consequently

z or $\frac{1}{y^n} = e^{n\int Pdx}\left\{-n\int Qe^{-n\int Pdx}dx + c\right\}$.

Cor. 2. It appears from the process in the article that it is not necessary to complete the fluent at the first integration; for c is so involved in z and x that in $y = xz$ it disappears, and becomes involved in c. If it were not to disappear, an equation of the first order would contain two arbitrary constants, which is impossible. (Ch. 3, art. 2.)

16. *The form* $dy + Py^2dx = Qdx$, *where* P *and* Q *are functions of* x, *is reducible to a linear equation of the second order.*

For substitute $z = e^{\int ydx}$; then, differentiating,

$$\frac{dz}{dx} = y^Pe^{\int ydx} = yPz$$

$$\frac{d^2z}{dx^2} = y^2P^2z + z\left\{P\frac{dy}{dx} + yP'\right\} \text{ if } P' = \frac{dP}{dx}$$

$$= y^2P^2z + z\left\{-P^2y^2 + PQ + yP'\right\}$$

$$= PQz + P'.zy = PQz + \frac{P'}{P}\frac{dz}{dx} \text{ or}$$

$\frac{d^2z}{dx^2} - \frac{P'}{P}\frac{dz}{dx} - PQz = 0$; an equation linear in z, which will be considered in the sixth chapter.

17. EXAMPLES.

Ex. 1. $dy + ydx = ax^ndx$.

Here $P = 1$, $Q = ax^n$ | $\therefore \int Qe^{\int Pdx}dx = \int ax^n e^x dx = ae^x(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots)$ (ch. 1, 38) $\therefore y = ce^{-x} + a(x^n - nx^{n-1} + \cdots n(n-1)x^{n-2} - \cdots)$ which will terminate if n be a positive integer.

If the fluent is to be corrected so that $c = 0$, we have $y = a(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots)$ an algebraick quantity; but if $y = 0$ when $x = 0$, y is always a transcendental.

Ex. 2. $dy = (a + bx + cy)dx$.

To reduce this to a simpler form, substitute $v = a + bx + cy$; and it becomes $\frac{dv - bdx}{c} = vdx$ or $dv - cvdx = bdx$

$$\therefore v = e^{cx}\left\{-\frac{b}{c}e^{-cx} + \frac{c}{c}\right\} = \frac{ce^{cx}}{c} - \frac{b}{c} \therefore \cdots$$

$$ce^{cx} = c(a + bx + cy) + b = bx + cy + a + \frac{b}{c}.$$

Ex. 3. $dy + \frac{nydx}{\sqrt{1+x^2}} = adx$.

$P = \frac{n}{\sqrt{1+x^2}}$, $Q = a$ | $\therefore \int Pdx = \int \frac{ndx}{\sqrt{1+x^2}} = l.(x + \sqrt{1+x^2})^n$ $\therefore e^{\int Pdx} = (x + \sqrt{1+x^2})^n$, and making x negative, $e^{-\int Pdx} = (\sqrt{1+x^2} - x)^n \therefore \cdots$

$$y = \sqrt{1+x^2} - x)^n a \int (x + \sqrt{1+x^2})^n dx.$$

To integrate $(x + \sqrt{1+x^2})^n dx = du$, substitute $v = x + \sqrt{1+x^2}$; then $v^2 - 2vx = 1$ or $x = \frac{v}{2} - \frac{1}{2v}$

and $dx = \frac{dv}{2} + \frac{dv}{2v^2} \therefore du = \frac{1}{2}(v^n dv + v^{n-2}dv) \cdots$

$$\therefore u = \tfrac{1}{2}\left\{\frac{v^{n+1}}{n+1} + \frac{v^{n-1}}{n-1}\right\} + c.$$

But $v^n = (x + \sqrt{1+x^2})^n = (\sqrt{1+x^2} - x)^{-n}$ or $\cdots$

$$(\sqrt{1+x^2} - x)^n = v^{-n} \therefore y = cv^{-n} + \frac{a}{2}\left\{\frac{v}{n+1} + \frac{v^{-1}}{n-1}\right\} \cdots$$

$$= c(\sqrt{1+x^2} - x)^n + \frac{a}{2}\left\{\frac{x + \sqrt{1+x^2}}{n+1} + \frac{\sqrt{1+x^2} - x}{n-1}\right\}$$

$= c(\sqrt{1+x^2}-x)^n + \frac{a}{n^2-1}(n\sqrt{1+x^2}-x).$

If $y=0$ when $x=0$, $c=-\frac{na}{n^2-1}$.

Ex. 4. $dy - xdy - aydx - bdx + bxdx = 0$ or

$dy - \frac{a}{1-x}ydx = bdx.$

To integrate this without referring to the formula, let $y = xz$; then $xdz + zdx - \frac{axzdx}{1-x} = bdx.$

Assume $dz - \frac{azdx}{1-x} = 0$ or $\frac{dz}{z} = \frac{adx}{1-x}$ $\therefore \frac{z}{c} = \frac{1}{(1-x)^a}$.

Also $dx = \frac{bdx}{z} = \frac{b}{c}(1-x)^a dx$ $\therefore x = \frac{-b}{c(a+1)}(1-x)^{a+1} + c'$;

and $y = xz = -\frac{b}{a+1}(1-x) + \frac{c}{(1-x)^a}$ . . . . . .

or $(a+1)y = c(1-x)^{-a} - b(1-x)$.

Ex. 5. $y^n dy - a^{n-1}xdy + b^{n-1}ydx = 0$ or

$dx - \left(\frac{a}{b}\right)^{n-1}\frac{xdy}{y} = -\left(\frac{y}{b}\right)^{n-1}dy.$

Here the equation is linear with respect to x; assume therefore $x = yz$; then

$ydz + zdy - \left(\frac{a}{b}\right)^{n-1}\frac{yzdy}{y} = -\left(\frac{y}{b}\right)^{n-1}dy$ $\therefore$ . . . . .

$\frac{dz}{z} = \left(\frac{a}{b}\right)^{n-1}\frac{dy}{y}$ $\therefore l\frac{z}{c} = \left(\frac{a}{b}\right)^{n-1}ly$ $\therefore$ . . . . . .

$\frac{z}{c} = y^r$ where $r = \left(\frac{a}{b}\right)^{n-1}$ or $z = cy^r$.

Also $dy = -\left(\frac{y}{b}\right)^{n-1}\frac{dy}{cy^r} = -\frac{y^{n-r-1}dy}{cb^{n-1}}$ $\therefore$ . . . . .

$y = -\frac{y^{n-r}}{(n-r)cb^{n-1}} + c$ $\therefore x = cy^r - \frac{y^n}{(n-r)b^{n-1}}$ . . . . .

$= cy^r - \frac{y^n}{nb^{n-1}-a^{n-1}}$.

Ex. 6. $\frac{dy}{dx} = \frac{a^2+x^2-y^2}{a^2}$. (Cambridge Problems, p. 413.)

Since $a^2(dy - dx) + (y^2 - x^2)dx = 0$, substitute $z = y - x$; then $a^2dz + (z^2 + 2xz)dx = 0$ or . . . . $\frac{a^2dz}{z^2} + \left(1 + \frac{2x}{z}\right)dx = 0$. To reduce this to a linear form, substitute $v = \frac{1}{z}$ $\therefore a^2dv - (1 + 2xv)dx = 0$ or . . . $dv - \frac{2x}{a^2}vdx = \frac{1}{a^2}dx \therefore v = e^{\frac{x^2}{a^2}}\left\{\frac{1}{a^2}\int e^{-\frac{x^2}{a}}dx + c\right\}$ which can be integrated only by developement.

Ex. 7. $dy + ydx = xy^3dx$.

This is of the form in cor. 1; hence z or $\frac{1}{y^2}$ . . . . . $= e^{2x}\left\{-2\int xe^{-2x}dx + c\right\} = ce^{2x} + e^{2x}(xe^{-2x} - \int e^{-2x}dx)\cdots = ce^{2x} + x + \frac{1}{2}$.

Ex. 8. $ay^mdy = bx^ndy - ayx^{n-1}dx$.

To reduce this to a linear form, let $v = x^n$; then $ay^mdy = bvdy - \frac{a}{n}ydv$ or $dv - \frac{nb}{a}\frac{vdy}{y} = -ny^{m-1}dy \cdots$ $\therefore v$ or $x^n = \frac{na}{nb-ma}y^m + cy^{\frac{nb}{a}}$.

18. PRAXIS.

1. $dy + \frac{n-1}{x}ydx = \frac{(n-1)dx}{x} \therefore y = 1 + \frac{c}{x^{n-1}}$.

2. $\frac{dy}{y} - \frac{dx}{x} = \frac{x^mdx}{an^{\frac{1}{2}}y} \therefore y = \frac{x^{m+1}}{man^{\frac{1}{2}}} + cx$.

3. $dy - \frac{yxdx}{1+x^2} = \frac{adx}{1+x^2} \therefore y = ax + c\sqrt{1+x^2}$.

4. $2ydx + (2a - x)dy = xdx \therefore y = x - a + \frac{(2a-x)^2}{c}$.

5. $dy + \frac{n-1-mx}{x(1-x)}ydx = \frac{(n-1)dx}{x(1-x)} \therefore$ . . . . . $y = \frac{(1-x)^{n-2n-1}}{x^{n-1}}\left\{(n-1)\int\frac{x^{n-2}dx}{(1-x)^{n-m}} + c\right\}$.

6. $ydy - \frac{y^2dx}{\sqrt{1+x^2}} = adx \therefore$ . . . . . $y^2 = c(x + \sqrt{1+x^2})^2 - \frac{2a}{3}(x + 2\sqrt{1+x^2})$.

7. $dy + 2ydx = y^2x^2dx \therefore \frac{1}{y} = ce^{x^2} + \frac{x^2}{2} + \frac{x}{2} + \frac{1}{4}$.

8. $aydy - by^2dx + cxdx = 0 \therefore y^2 = ce^{\frac{2bx}{a}} + \frac{cx}{b} + \frac{ac}{2b^2}$.

19. *Required to investigate the cases in which* $dy + by^2dx = ax^mdx$ *is integrable.*

When $m = 0$, $dy + by^2dx = adx$ and $dx = \frac{dy}{a - by^2}$; a transcendental form.

Next, to find under what values of m it may be rendered homogeneous.

Substitute $y = z^k$; then $kz^{k-1}dz + bz^{2k}dx = ax^mdx$; and that this may be homogeneous, we must have $k - 1 = 2k = m$, or $k = -1$ and $m = -2$; whence $dy + by^2dx = ax^{-2}dx$ is the only form which can be rendered homogeneous; in which case, substituting $y = z^{-1}$, it becomes

$-\frac{dz}{z^2} + \frac{bdx}{z^2} = \frac{adx}{x^2}$, which can be integrated as in art. 10.

To ascertain whether there are any other cases which are integrable by a substitution of the form $y = ex^p + x^qz$; we have $dy = (pex^{p-1} + qx^{q-1}z)dx + x^qdz$ and $y^2dx = (e^2x^{2p} + 2ex^{p+q}z + x^{2q}z^2)dx$, and consequently $x^qdz + (qx^{q-1} + 2bex^{p+q} + bx^{2q}z)zdx + (pex^{p-1} + be^2x^{2p})dx = ax^mdx$.

This equation contains only three terms, if we have $p - 1 = 2p$, $pe + be^2 = 0$, $q - 1 = p + q$, and $q + 2be = 0$. The first and third conditions agree in giving $p = -1$; and from the second and fourth, there results $e = \frac{1}{b}$, $\cdots$ $q = -2$; wherefore $y = \frac{1}{bx} + \frac{z}{x^2}$; and this substituted in the proposed equation, transforms it into

$\frac{dz}{x^2} + \frac{bz^2dx}{x^4} = ax^mdx$; or $dz + \frac{bz^2}{x^2}dx = ax^{m+2}dx$. ($\alpha$)

Hence, when $m = -2$, the proposed equation may be transformed into one which is homogeneous, as has been already shown.

Also, when $m = -4$, the variables may be separated; for (α) becomes $\frac{dz}{a - bz^2} = \frac{dx}{x^2}$.

Further, if in (α) we substitute $z = \frac{1}{y'}$, there will result $-dy' + \frac{bdx}{x^2} = ay'^2x^{m+2}dx$; or $dy' + ay'^2x^{m+2}dx = \frac{bdx}{x^2}$; and supposing $x^{m+2}dx = \frac{dx'}{m+3}$, which gives $x^{m+3} = x'$, and consequently $\frac{dx}{x^2} = \frac{1}{m+3} \cdot x'^{-\frac{m+4}{m+3}}dx'$, the equation becomes $dy' + \frac{a}{m+3}y'^2dx' = \frac{b}{m+3}x'^{-\frac{m+4}{m+3}}dx'$; in this, for the sake of brevity, substitute $b' = \frac{a}{m+3}$, $a' = \frac{b}{m+3}$ and $m' = -\frac{m+4}{m+3}$, and there will result $dy' + b'y'^2dx' = a'x'^{m'}dx'$; an equation which is of the same form as the proposed, and consequently is integrable when $m' = 0$ or -2 or -4.

Similarly it may be shown that this last equation may be transformed into one of the form $dy'' + b''y''^2dx'' = a''x''^{m''}dx''$ where $y'' = \frac{1}{z'}$, z' being determined from the equation $y' = \frac{1}{b'x'} + \frac{z'}{x'^2}$; also $b'' = \frac{a'}{m'+3}$, $a'' = \frac{b'}{m'+3}$, $x'' = x'^{m'+3}$, and $m'' = -\frac{m'+4}{m'+3}$. This equation is integrable when $m'' = 0$ or -2 or -4.

By repeating these transformations, it will be seen that the original equation may be transformed into one that is integrable, when either m' which $= -\frac{m+4}{m+3}$, or m'' which $= -\frac{m'+4}{m'+3} = -\frac{3m+8}{2m+5}$, or m''' which $= -\frac{m''+4}{m''+3}$ . . . $= -\frac{5m+12}{3m+7}$, or &c. is equal to 0 or -2 or -4.

The values of m which satisfy these conditions are -4, $-\frac{8}{3}, -\frac{12}{5}, -\frac{16}{7}$, $-$ &c. and are included in the form

$-\frac{4i}{2i-1}$ where i is any positive integer.

Next transform the *original equation* by substituting $y=\frac{1}{y'}$ and $x^{m+1}=x'$ and the result will be of the form $dy'+b'y'^{\circ}dx'=a'x'^{m'}dx'$ where $m'=-\frac{m}{m+1}$; which therefore is integrable when $-\frac{m}{m+1}=-\frac{4i}{2i-1}$ or $m=-\frac{4i}{2i+1}$; and consequently the proposed form is integrable when $m=-\frac{4i}{2i\pm1}$.

Cor. 1. The equation $dy+ay^{\circ}x^{p}dx=bx^{q}dx$, the form considered by Riccati, may be reduced to the form of the article by substituting $x^{p}dx=dv$.

Cor. 2. When $m=2$, $i=\infty$; or it requires an infinite number of transformations of the form $y=\frac{1}{bx}+\frac{z}{x^2}$ to render the equation homogeneous.

Ex. 1. $dy+y^{\circ}dx=a^{\circ}x^{-4}dx$.

Here $m=-4$, and the variables are separable.

Substitute $y=\frac{1}{x}+\frac{z}{x^2}$; then

$$-\frac{dx}{x^2}+\frac{dz}{x^2}-\frac{2zdx}{x^3}+\left\{\frac{1}{x}+\frac{z}{x^2}\right\}^2dx=a^{\circ}x^{-4}dx \quad . \quad .$$

or $\frac{dz}{x^2}+\frac{z^2dx}{x^4}=\frac{a^{\circ}dx}{x^4}\therefore\frac{dx}{x^2}=\frac{dz}{a^2-z^2}$; and changing the signs that the system of logarithms may be positive (vol. i. c. 2, 16),

$$-\frac{dx}{x^2}=\frac{dz}{z^2-a^2}\therefore\frac{1}{x}=\frac{1}{2a}l\frac{c(z-a)}{z+a} \text{ or } e^{\frac{2a}{x}}=\frac{c(z-a)}{z+a}\therefore\ldots$$

$$c=e^{\frac{2a}{x}}\cdot\frac{(z+a)}{z-a}=e^{\frac{2a}{x}}\cdot\frac{x^2y-x+a}{x^2y-x-a}.$$

Ex. 2. $dy+y^{\circ}dx=-a^{\circ}x^{-4}dx$.

Substituting as in ex. 1, we have $-\frac{dx}{x^2}=\frac{dz}{a^2+z^2}\therefore \quad . \quad . \quad .$

$$\frac{a}{x}+c=\tan.^{-1}\frac{x^2y-x}{a}.$$

Ex. 3. $dy + y^m dx = a^2 x^{-\frac{8}{3}} dx$, where m is of the form $-\frac{4i}{2i-1}$.

Here equation (α) is $dz + \frac{z^2 dx}{x^2} = a^2 x^{-\frac{2}{3}} dx$; in which substitute $y' = \frac{1}{z}$ $\therefore -\frac{dy'}{y'^2} + \frac{dx}{y'^2 x^2} = a^2 x^{-\frac{8}{3}} dx$ . . . .

or $dy' + y'^2 . a^2 x^{-\frac{2}{3}} dx = \frac{dx}{x^2}$.

Let $dx' = a^2 x^{-\frac{2}{3}} dx$; then $x' = 3a^2 x^{\frac{1}{3}}$ or . . . . . $\frac{1}{x} = \frac{27a^6}{x'^3}$ $\therefore \frac{dx}{x^2} = 81a^6 x'^{-4} dx$; and by substitution ($\alpha$) becomes $dy' + y'^2 dx' = 81a^6 x'^{-4} dx$, whose fluent (ex. 1) is $c = e^{\frac{18a^3}{x'}} \cdot \frac{x'^2 y' - x' + 9a^3}{x'^2 y' - x' - 9a^3} = e^{6ax^{-\frac{1}{3}}} \cdot \frac{3a^2 x^{\frac{2}{3}} - x^{\frac{1}{3}} z + 3az}{3a^2 x^{\frac{2}{3}} - x^{\frac{1}{3}} z - 3az}$, where $z = x^2 y - x$.

Ex. 4. $dy - y^2 dx = a^2 x^{-\frac{8}{3}} dx$.

Here substitute $y = -\frac{1}{x} + \frac{z}{x^2}$; and the equation becomes $dz - \frac{z^2 dx}{x^2} = a^2 x^{-\frac{2}{3}} dx$; in which substitute $y' = \frac{1}{z}$;

$\therefore -\frac{dy'}{y'^2} - \frac{dx}{y'^2 x^2} = a^2 x^{-\frac{2}{3}} dx$ or $dy' + y'^2 . a^2 x^{-\frac{2}{3}} dx = -\frac{dx}{x^2}$.

Let $dx' = a^2 x^{-\frac{2}{3}} dx$; then, as in ex. 3, we have $dy' + y'^2 dx' = -81a^6 x'^{-4} dx$, whose fluent (ex. 2) is . . . .

$\frac{9a^3}{x'} + c = \tan.^{-1} \frac{x'^2 y' - x'}{9a^3}$ or . . . . . .

$\frac{3a}{x^{\frac{1}{3}}} + c = \tan.^{-1} \frac{3a^2 - x^{\frac{5}{3}} y - x^{\frac{2}{3}}}{3ax^{\frac{1}{3}}(xy+1)}$.

Ex. 5. $dy + y^2 dx = a^2 x^{-\frac{4}{3}} dx$.

Here m is of the form $-\frac{4i}{2i+1}$; whence substitute in the

proposed $y' = \frac{1}{y}$; then $-\frac{dy'}{y'^2} + \frac{dx}{y'^2} = a^2 x^{-\frac{4}{3}} dx$ or . . .

$dy' + y'^2 \,.\, a^2 x^{-\frac{4}{3}} dx = dx.$

Let $dx' = a^2 x^{-\frac{4}{3}} dx \therefore x' = -3a^2 x^{-\frac{1}{3}}$ or $x = -\frac{27a^6}{x'^3}$

$\therefore dx = \frac{81a^6}{x'^4}$; and the proposed becomes

$dy' + y'^2 dx' = 81a^6 x'^{-4} dx$; whose fluent (ex. 1) is

$$c = e^{\frac{18a^3}{x'}} \cdot \frac{x'^2 y' - x' + 9a^3}{x'^2 y' - x' - 9a^3} = e^{6a^3 x'^{\frac{1}{3}}} \cdot \frac{3a^2 - x'^{\frac{1}{3}} y + 3a x'^{\frac{2}{3}} y}{3a^2 - x'^{\frac{1}{3}} y - 3a x'^{\frac{2}{3}} y}.$$

Ex. 6. $dy + y^2 dx = a^2 x^{-\frac{8}{5}} dx.$

Substitute $y' = \frac{1}{y}$; then $dy' + y'^2 \,.\, a^2 x^{-\frac{8}{5}} dx = dx.$

Let $dx' = a^2 x^{-\frac{8}{5}} dx$; then $x' = -\frac{5a^2 x^{-\frac{3}{5}}}{3}$ and

$x = -\left(\frac{3x'}{5a^2}\right)^{-\frac{5}{3}}$; and the equation becomes of the form

$dy' + y'^2 dx' = a' x'^{-\frac{8}{3}} dx$, which coincides with ex. 3.

Ex. 7. $dy + by^2 x^n dx = ax^m dx.$

Let $\frac{dx'}{n+1} = x^n dx$; then $x' = x^{n+1}$ and $x^{m+1} = x'^{\frac{m+1}{n+1}}$, $\therefore \cdots$

$x^m dx = \frac{1}{n+1} \cdot x'^{\frac{m-n}{n+1}} dx'$; and the proposed becomes

$dy + b' y^2 dx' = a' x'^{\frac{m-n}{n+1}} dx'$; which is integrable when $\frac{m-n}{n+1}$

is of the form $-\frac{4i}{2i \pm 1}$.

20. *Required to investigate the cases in which* . . . $dy + by^q x^n dx + ay^p x^m dx = 0$ *is integrable.*

(1.) If $q = n = p = m = 0$ } the equation is homoge-
(2.) If $q + n = p + m = 0$ } neous.
(3.) If $q = p$ }
(4.) If $n = m$ } the variables may be separated.

To find the conditions that the equation may be rendered homogeneous, substitute $y = z^k$; wherefore $kz^{k-1}dz + bz^{qk}x^n dx + ay^{pk}x^m dx = 0$, which is homogeneous when $k - 1 = qk + n = pk + m$, from which $k = \frac{n+1}{1-q} = \frac{m+1}{1-p}$; or the equation of condition is $(1-p)(n+1) = (1-q)(m+1)$. If this obtains, the proposed equation will become homogeneous by substituting $y = z^{\frac{n+1}{1-q}}$ or $z^{\frac{m+1}{1-p}}$.

If $q = p = 1$, the substitution fails; but the equation is now included in case (3).

Similarly, the conditions may be obtained for equations containing four or more terms; and it may be shown that $dy + (ax^m y^n + bx^{m'}y^{n'} + cx^{m''}y^{n''} + \ldots)dx = 0$ may be rendered homogeneous when $\frac{m+1}{1-n} = \frac{m'+1}{1-n'} = \frac{m''+1}{1-n''} = \cdots$ by assuming $k = \frac{m+1}{1-n}$.

Ex. 1. $ay^3 x dx + by^2 x^{\frac{1}{2}} dx = cx dy$; or . . . . . $dy - \frac{b}{c}y^2 x^{-\frac{1}{2}} dx - \frac{a}{c}y^3 dx = 0$.

Here $\frac{n+1}{1-q} = -\frac{1}{2} = \frac{m+1}{1-p}$; $\therefore$ substitute $y = z^{-\frac{1}{2}}$, and there results $az^{-\frac{3}{2}}x dx + bz^{-1}x^{\frac{1}{2}}dx = -\frac{cxz^{-\frac{3}{2}}dz}{2}$; which is homogeneous.

Ex. 2. $ay^{\frac{1}{3}}x^{\frac{11}{6}}dx - 6x^3 y^{-1}dy = cx^2 y dy$.

Substitute $x = z^k$; then $kay^{\frac{1}{3}}x^{\frac{17k}{6}-1}dz - 6z^{3k-1}y dy \cdots = cz^{2k}y dy$; which becomes homogeneous if $k = 2$.

Ex. 3. $ay^2x^2dx + bdx + cyxdx + ex^4y^2dy = 0$. Substitute $y = z^{-1}$.

Ex. 4. $x^{\frac{1}{2}}dx + y^{\frac{4}{3}}dx + x^{\frac{3}{4}}y dy = y^3 dy$; $k = \frac{3}{8}$.

Ex. 5. $dy + \left\{ ax^m y^n + \frac{by}{x} \right\} dx = 0$.

Substitute $y = z^{\frac{m+1}{1-n}}$ ∴ . . . .

$$\frac{m+1}{1-n} z^{\frac{m+n}{1-n}} dz + \left\{ ax^m z^{\frac{n(m+1)}{1-n}} + \frac{bz^{\frac{m+1}{1-n}}}{x} \right\} dx = 0,$$

or $\frac{m+1}{1-n} dz + \left\{ ax^m z^{-m} + \frac{bz}{x} \right\} dx = 0$, which is homogeneous and integrable.

Ex. 6. $dy + \left\{ ax^m y^n + bx^{\frac{m-n+2}{n-1}} y^2 + cx^{\frac{2m-n+3}{n-1}} y^3 \right\} dx = 0$ may be rendered homogeneous by substituting $y = z^{\frac{m+1}{1-n}}$.

21. *On finding a factor which renders an equation an exact fluxion.*

An equation which is the immediate derivative of $u = \text{F}(x, y) = 0$ is an exact fluxion. Equations of this kind seldom occur in calculation; those which we generally meet with are such as have been derived from their primitive by the elimination of a constant, and these are not exact fluxions, but may be rendered so by multiplying by some factor.

Thus, let $y - ax = 0$; differentiating, $dy - adx = 0$, which is an exact fluxion; but if we eliminate a, the result is $ydx - xdy = 0$, which may be rendered an exact fluxion by multiplying by $\frac{1}{y^2}$.

In general, all equations of the first degree which are not exact fluxions must have resulted from eliminating a constant from the primitive and its immediate derivative. It appears from the process of elimination that the resulting equation ceases to be an exact fluxion in consequence of the disappearance of a factor; if then this factor can be restored, the equation becomes *integrable per se.*

Thus, let the primitive be $\text{P} + c\text{Q} = 0$ where P and Q are functions of the variables; then its immediate derivative is $d\text{P} + cd\text{Q} = 0$, and eliminating c, $\text{Q}d\text{P} - \text{P}d\text{Q} = 0$. Now this must be the same equation as that which would arise from solving the primitive with respect to c, and then differentiating; in which case the primitive is $c = -\frac{\text{P}}{\text{Q}}$, and

the equation is $o = \frac{Qdp - pdQ}{Q^2}$ an exact fluxion; which shows that $Qdp - pdQ$ ceases to be an exact fluxion in consequence of the disappearance of the factor $\frac{1}{Q^2}$.

If Qdp and pdQ have a common measure, this may also disappear in the process of elimination; but the above proposition holds good, viz. that the function necessary to render $Qdp - pdQ$ an exact fluxion is under the form of a *factor*.

We shall give a more direct demonstration of this proposition in the following article.

22. *An equation of the first order between two variables may be always rendered an exact fluxion by the introduction of a factor.*

By solving the proposed equation with respect to p, it may be made to take the form $p + f(x, y) = 0$. Let the primitive, when solved with respect to its arbitrary constant c, be $F(x, y) = c =$, by substitution, u where u is a function of (x, y) not containing c.

Now, differentiating $u = F(x, y)$, we have

$\frac{du}{dx} = 0 = \frac{du}{dx} + \frac{du}{dy}p = u' + u_1 p$ or $p + \frac{u'}{u_1} = 0$; an equation which is identical with $p + f(x, y) = 0$ (ch. 3, art. 6);

or $p + f(x, y) = p + \frac{u'}{u_1} = \frac{1}{u_1}(u' + u_1 p) = \frac{1}{u_1} \cdot \frac{du}{dx}$; $\cdots$

wherefore $\left\{ p + f(x, y) \right\} u_1 = \frac{du}{dx}$; or the proposed equation, when reduced to the form $p + f(x, y) = 0$, may be rendered an exact fluxion by means of the factor $\frac{du}{dy} = \frac{dc}{dy}$, the fluxional coefficient of c in $c = F(x, y)$.

Cor. 1. If the primitive be known, and it can be solved with respect to its constant, the requisite factor can be found.

Cor. 2. If the process of separating the variables be known, the factor proper to render the equation an exact fluxion can be found.

For the proposed equation is not an exact fluxion in con-

sequence of the disappearance of a factor; if then by any process we can separate the variables, and thus render it an exact fluxion, the factor will reappear.

Ex. 1. $y^2dx + xydx - x^2dy = 0$.

Since the equation is homogeneous, the variables are separated by substituting $y = xz$; and it becomes

$x^2(z^2dx - xdz) = 0$, or $x^3z^2\left\{\frac{dx}{x} - \frac{dz}{z^2}\right\} = 0$; or the requisite factor is $\frac{1}{x^3z^2} = \frac{1}{y^2x}$. Multiplying by this, the equation becomes $\frac{dx}{x} + \frac{dx}{y} - \frac{xdy}{y^2} = 0$, which is an exact fluxion.

Ex. 2. $dy + y^2dx - a^2x^{-4}dx = 0$.

Substituting $y = \frac{1}{x} + \frac{z}{x^2}$, the equation becomes

$\frac{dz}{x^2} + \frac{z^2dx}{x^4} - \frac{a^2dx}{x^4} = 0$; or $\frac{z^2-a^2}{x^2}\left\{\frac{dz}{z^2-a^2} + \frac{dx}{x^2}\right\} = 0$, and the requisite factor is $\frac{x^2}{z^2-a^2}$ or $\frac{x^2}{(x^2y-x)^2-a^2}$.

23. *Euler's Criterion of Integrability.*

Let $du = \text{M}dx + \text{N}dy$ where M and N are functions of (x, y); then, when du is an exact fluxion, we have $\frac{du}{dx} = \text{M}$ and $\frac{du}{dy} = \text{N}$; and, differentiating partially, $\frac{d\text{M}}{dy} = \frac{d^2u}{dxdy}$ = (vol. i. ch. 4, 16) $\frac{d^2u}{dydx} = \frac{d\text{N}}{dx}$; or the condition in order that $\text{M}dx + \text{N}dy$ shall be an exact fluxion is that $\frac{d\text{M}}{dy} = \frac{d\text{N}}{dx}$.

Cor. 1. A formula in which the variables are separate is an exact fluxion.

For, let $du = \text{X}dx + \text{Y}dy$; then $\frac{d\text{X}}{dy} = 0 = \frac{d\text{Y}}{dx}$.

Cor. 2. $\frac{d\int \text{M}dx}{dy} = \int \frac{d\text{M}}{dy}dx$.

For, let $u = \int \text{M}dx$; then $\frac{du}{dx} = \text{M}$, and $\frac{d^2u}{dxdy}$, which

$= \frac{d^2u}{dydx}$, $= \frac{d\text{M}}{dy}$; wherefore, integrating with respect to x,

$\frac{du}{dy}$ or $\frac{d\int \text{M}dx}{dy} = \int \frac{d\text{M}}{dy}dx$

The theorem demonstrated in this corollary is called Leibnitz's from the name of its inventor.

Ex. 1. $(2x - y)dx - xdy$ is an exact fluxion.

For $\frac{d\text{M}}{dy} = -1 = \frac{d\text{N}}{dx}$.

Ex. 2. $ydx - xdy$ is not an exact fluxion.

For $\frac{d\text{M}}{dy} = 1$ and $\frac{d\text{N}}{dx} = -1$.

Ex. 3. $\frac{dx}{x} + \frac{y^2dx}{x^3} - \frac{ydy}{x^2} + \frac{ydx - xdy}{x^3}\sqrt{x^2+y^2} + \frac{dy}{2y}$ is an exact fluxion.

This example may be put under the form

$$\left\{\frac{1}{x} + \frac{y^2}{x^3} + \frac{y\sqrt{x^2+y^2}}{x^3}\right\}dx + \left\{\frac{1}{2y} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right\}dy,$$

$$\therefore \frac{d\text{M}}{dy} = \frac{2y+\sqrt{x^2+y^2}}{x^3} + \frac{y^2}{x^3\sqrt{x^2+y^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$= \frac{2y\sqrt{x^2+y^2}+x^2+2y^2}{x^3\sqrt{x^2+y^2}}, \text{ and}$$

$$\frac{d\text{N}}{dx} = -\frac{1}{x^2}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{2}{x^3}(y+\sqrt{x^2+y^2}) = \frac{d\text{M}}{dy}.$$

24. *Required to integrate a function of two variables which is an exact fluxion.*

Let $du = \text{M}dx + \text{N}dy$ be an exact fluxion; then

$\frac{du}{dx} = \text{M}$ and $\frac{du}{dy} = \text{N}$.

Now integrating on the supposition that y is constant, we have $u = \int \text{M}dx + \text{Y}\,(\alpha)$, where the correction Y is a sole function of y.

Next differentiate (α) partially with respect to y; then

$\frac{du}{dy}$ or $\text{N} = \frac{d.\int \text{M}dx}{dy} + \frac{d\text{Y}}{dy}$ or $d\text{Y} = \left\{\text{N} - \frac{d.\int \text{M}dx}{dy}\right\}dy$ and

$\text{Y} = \int \left\{ \text{N} - \frac{d.\int \text{M}dx}{dy} \right\} dy$. Substitute this value of Y in (α), and there results $u = \int \text{M}dx + \int \left\{ \text{N} - \frac{d.\int \text{M}dx}{dy} \right\} dy$.

Cor. 1. Since $u = \int \text{M}dx + \int \left\{ \text{N} - \frac{d.\int \text{M}dx}{dy} \right\} dy$, and also $= \int \text{M}dx + \text{Y}$; therefore $\int \left\{ \text{N} - \frac{d\int \text{M}dx}{dy} \right\} dy = \text{Y}$, and consequently $\text{N} - \frac{d\int \text{M}dx}{dy}$ does not contain x. Hence

Rule 1. 'First integrate one of the terms as $\text{M}dx$ on the supposition that y is constant; to this add the fluent of as many of the terms in $\text{N}dy$ as do not contain x.'

In the application of this rule, the student must be careful to observe whether the terms in $\text{N}dy$ which he neglects may not be decomposable into others, some of which may contain only y. If this should be the case, the rule will give an erroneous result; because these terms which contain only y, when integrated, form part of the correction and may not be neglected. Thus, $\frac{x dy}{y\sqrt{x^2+y^2}}$ where N contains x is decomposable into $dy \left\{ \frac{1}{y} - \frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})} \right\}$; and if we integrate according to the rule, we neglect ly which forms part of the fluent. If there is any doubt in respect to the nature of the coefficient N and none in respect to M, we may first integrate $\text{N}dy$ on the supposition that x is constant, and to this add the fluent of as many of the terms in $\text{M}dx$ as do not contain y.

When these two processes give the same result, the true value of u is obtained. Or the result may be verified by ascertaining whether it gives the proposed value of $\frac{du}{dy}$. If both the coefficients be doubtful, we should integrate by the following safer rule.

Rule 2. 'First integrate du with respect to x, add the correction Y a function of y; differentiate the result with respect to y. Equate the resulting value of $\frac{du}{dy}$ with its pro-

posed value, by which $\frac{dY}{dy}$, and consequently Y may be found.'

Cor. 2. To verify the equation of condition, let $v = \int M dx$; then, since $N - \frac{dv}{dy}$ does not contain x, we have, differentiating with respect to x, $\frac{dN}{dx} - \frac{d^2v}{dydx} = 0$, or $\frac{dN}{dx} = \frac{d^2v}{dydx} = \frac{d^2v}{dxdy} = \frac{d.\frac{dv}{dx}}{dy} = \frac{dM}{dy}$, which is the equation of condition.

Cor. 3. Since $u = \int M dx + Y$ and also $= \int N dy + X$, therefore $\int M dx + Y = \int N dy + X$, and $\int M dx - \int N dy = X - Y$; or the difference of the fluents integrated partially with respect to x and to y may be always separated into two parts, each of which is an *explicit* function.

25. In the following examples, having previously ascertained that the equation of condition obtains, we may either integrate by the rules; or substitute for M and N in the formula $u = \int M dx + \int \left\{ N - \frac{d\int M dx}{dy} \right\} dy$.

Ex. 1. $du = \frac{dx}{\sqrt{1+x^2}} + a dx + 2by dy$.

$du = \left\{ \frac{1}{\sqrt{1+x^2}} + a \right\} dx + 2by dy$. Here $\frac{dM}{dy} = 0 = \frac{dN}{dx}$

$\therefore$ by the rule, $u = l(x + \sqrt{1+x^2}) + ax + by^2$; which is the function which arises from integrating each term separately.

Ex. 2. $du = 2axy dx + ax^2 dy - y^3 dx - 3xy^2 dy$.
$du = (2axy - y^3) dx + (ax^2 - 3xy^2) dy$ $\therefore$ (rule 1),
$u = ax^2 y - y^3 x$.

When du is homogeneous and an exact fluxion as in this example, u may be found by changing dx and dy into x and y, and dividing by the dimension of the resulting function (art. 7). This method fails when du is of -1 dimensions.

Ex. 3. $du = \frac{dx}{x} + \frac{y^2 dx}{x^3} - \frac{ydy}{x^2} + \frac{ydx - xdy}{x^3} \sqrt{x^2 + y^2} + \frac{dy}{2y}$ (art. 23, ex. 3).

$du = \left\{ \frac{1}{x} + \frac{y^2}{x^3} + \frac{y\sqrt{x^2+y^2}}{x^3} \right\} dx + \left\{ \frac{1}{2y} - \frac{y}{x^2} - \frac{\sqrt{x^2+y^2}}{x^2} \right\} dy$

$$\therefore u = lx - \frac{y^2}{2x^2} + yf\frac{\sqrt{x^2+y^2}}{x^3}dx + \tfrac{1}{2}ly$$

$$= \tfrac{1}{2}l.x^2y - \frac{y^2}{2x^2} + y\left\{-\frac{\sqrt{x^2+y^2}}{2x^2} + \frac{1}{2y}l.\frac{\sqrt{x^2+y^2}-y}{x}\right\}$$

$$= \tfrac{1}{2}l.xy(\sqrt{x^2+y^2}-y) - \frac{y}{2x^2}(\sqrt{x^2+y^2}+y).$$

Ex. 4. $du = \dfrac{2xdy-2ydx}{(x-y)^2}$ $\therefore u = \dfrac{2y}{x-y} + c.$

By changing the value of c, the same fluent may appear under forms which, though apparently different, are essentially the same; thus substituting in this example $c+2=c$, we have $u = \dfrac{2x}{x-y} + c$; substituting $c+1=c$, $u = \dfrac{x+y}{x-y} + c.$

Ex. 5. $du = \dfrac{ydy+xdx-2ydx}{(y-x)^2}.$

$$u = f\frac{xdx-2ydx}{x^2-2yx+y^2} + \text{Y} = l(y-x) - f\frac{ydx}{(y-x)^2} + c, \text{ for } d\text{Y} = 0,$$

$$= l\frac{y-x}{c} - \frac{y}{y-x}.$$

If we first integrate with respect to y,

$u = l(y-x) - \dfrac{x}{y-x} + c$; which is the same result as before, the correction being increased by unity.

Ex. 6. $du = \dfrac{xdy-ydx}{y^2+x^2}$ $\therefore u = \text{tan.}^{-1}\dfrac{y}{x} + c.$

Ex. 7. $du = \dfrac{dx}{\sqrt{x^2+y^2}} - \dfrac{xdy}{y\sqrt{x^2+y^2}}.$

Here it is safer to integrate Ndy; whence we have

$$u = -f\frac{xdy}{y\sqrt{x^2+y^2}} = -l\frac{\sqrt{x^2+y^2}-x}{y} \quad \ldots\ldots$$

$$= l\frac{\sqrt{x^2+y^2}+x}{y} + c.$$

Or thus. Integrating partially,

$u = l(x + \sqrt{x^2+y^2}) + \text{Y} \therefore \frac{du}{dy} = \frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})} + \frac{d\text{Y}}{dy}$ which $\therefore = -\frac{x}{y\sqrt{x^2+y^2}}$, or $-\frac{d\text{Y}}{dy} = \frac{1}{y} \therefore \text{Y} = l\frac{1}{y}$ and $u = l\frac{x+\sqrt{x^2+y^2}}{y}$.

26. PRAXIS.

1. $du = (3x^2 + 2bxy - 3y^2)dx + (bx^2 - bxy + 3ay^2)dy$
$\therefore u = x^3 + bx^2y - 3y^2x + ay^3 + c.$

2. $du = (2y^2x + 3y^3)dx + (2x^2y + 9xy^2 + 8y^3)dy \therefore$
$u = y^2x^2 + 3y^3x + 2y^4 + c.$

3. $du = \frac{ydx - xdy}{(x+y)^2} \therefore u = \frac{x}{x+y}.$

4. $du = \frac{ydx - xdy}{y\sqrt{y^2-x^2}} \therefore u = \sin.^{-1}\frac{x}{y}$, or $= \text{cosec.}^{-1}\frac{y}{x}.$

5. $du = \frac{ydx - xdy}{y\sqrt{x^2-y^2}} \therefore u = l \cdot \frac{x+\sqrt{x^2-y^2}}{y}.$

6. $du = \frac{ydx - xdy}{(x+y)\sqrt{xy-y^2}} \therefore u = \sqrt{2}\sin.^{-1}\sqrt{\frac{x-y}{x+y}}.$

7. $du = \frac{dx(x+\sqrt{x^2+y^2}) + ydy}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})} \therefore u = l.c(x+\sqrt{x^2+y^2}).$

8. $du = \frac{y^2-xy}{(x^2+y^2)^{\frac{3}{2}}}dx + \frac{x^2-xy}{(x^2+y^2)^{\frac{3}{2}}}dy \therefore u = \frac{x+y}{\sqrt{x^2+y^2}}.$

9. $du = \frac{x^3dy + y^2dx}{(x^2+y^2)\sqrt{x^2+y^2-x^2y^2}} \therefore \ldots\ldots$
$u = \cos.^{-1}\sqrt{1 - \frac{x^2y^2}{x^2+y^2}}.$

10. $(ax + by + c)dx + (bx + ey + f)dy = 0$
$\therefore c = \frac{1}{2}ax^2 + bxy + cx + \frac{1}{2}ey^2 + fy.$
and $\int \text{M}dx - \int \text{N}dy = (\frac{1}{2}ax^2 + cx) - (\frac{1}{2}ey^2 + fy).$

11. $\frac{dx}{\sqrt{x^2+y^2}} + \frac{dy}{y}\left\{1 - \frac{x}{\sqrt{x^2+y^2}}\right\} = 0$
$\therefore c = x + \sqrt{x^2+y^2}.$

12. $\dfrac{a(xdx+ydy)}{\sqrt{x^2+y^2}} + \dfrac{ydx-xdy}{x^2+y^2} + 3by^2dy = 0$

$\therefore c = a\sqrt{x^2+y^2} + \text{tan.}^{-1}\dfrac{x}{y} + by^3.$

13. $\text{sin.}ydx + x\text{cos.}ydy + \text{sin.}xdy + y\text{cos.}xdx = 0$
$\therefore x\text{sin.}y + y\text{sin.}x = c.$

27. It has been shown (ch. 3, 13), that to every equation of the first order there belongs a primitive containing one arbitrary constant. It also appears that the primitive can be found, if the proposed equation is an exact fluxion: if it is not, there is some factor which will make it one; and could this be found in all cases, the inverse calculus, so far as relates to equations of the first order, would be perfect. We shall therefore proceed to consider the cases in which the nature of the factor can be ascertained, and shall add a few particular forms rather as specimens of Euler's manner of treating the subject than as being of any great utility.

28. *Required to find the factor which will render a form of the first order and degree an exact fluxion.*

Let the proposed form be $\text{P}dx + \text{Q}dy$, and when multiplied by the factor z, let it become $\text{M}dx + \text{N}dy$ an exact fluxion; then $\text{M} = \text{P}z$ and $\text{N} = \text{Q}z$.

Since $\text{M}dx + \text{N}dy$ is an exact fluxion (art. 23),

$\dfrac{d\text{M}}{dy} = \dfrac{d\text{N}}{dx}$; in which, substituting for M and N, Pz and Qz, we have $\dfrac{\text{P}dz+zd\text{P}}{dy} = \dfrac{\text{Q}dz+zd\text{Q}}{dx}$, or . . . . . .

$$\text{P}\frac{dz}{dy} - \text{Q}\frac{dz}{dx} + \left\{\frac{d\text{P}}{dy} - \frac{d\text{Q}}{dx}\right\}z = 0 \quad (\alpha).$$

The proposed, if it be an equation, may be put under the form $\text{P}dx + dy = 0$; in which case (α) becomes

$$\text{P}\frac{dz}{dy} - \frac{dz}{dx} + z\frac{d\text{P}}{dy} = 0.$$

If we could integrate this equation, and thus obtain the value of z; we should be enabled to render the proposed form an exact fluxion, and to integrate it. But the equation, except in certain cases, is obviously more difficult of integration than the proposed; since it contains three variables and has two fluxional coefficients.

z can always be found in the following cases.

Case 1. Let z be a constant quantity.

Here $\frac{dz}{dy} = 0 = \frac{dz}{dx}$, and (α) becomes $\frac{dP}{dy} - \frac{dQ}{dx} = 0$;

which shows that the proposed form is an exact fluxion, and no factor is necessary.

Case 2. Let $Pdx + Qdy$ be homogeneous.

Let $du = Mdx + Ndy = Pzdx + Qzdy$; and let P or Q be of m, and z, which is also homogeneous, of n dimensions; then since the degree of u is higher by unity than that of Pz or Qz, we have (art. 7), $Pzx + Qzy = (m + n + 1)u$; and dividing $Pzdx + Qzdy = du$ by this, we have

$\frac{Pdx+Qdy}{Px+Qy} = \frac{du}{u} \cdot \frac{1}{m+n+1}$; or $\frac{1}{Px+Qy}$ is a factor which will render $Pdx + Qdy$ an exact fluxion.

This fails when $Px + Qy = 0$.

Otherwise, by means of separating the variables.

Substitute as in art. 10, $\frac{y}{x} = z$, and let $P = x^n Fz$ and $Q = x^n fz$; then the proposed equation, eliminating $dy = zdx + xdz$, becomes $x^n Fzdx + x^n fz\{zdx + xdz\} = 0$, or $x^n\{Fz + zfz\}dx + x^{n+1}fzdz = 0$; an equation which is rendered an exact fluxion by the factor $\frac{1}{x^{n+1}\{Fz+zfz\}}$; which $= \frac{1}{Px+Qy}$.

Case 3. Let z be a sole function of one of the variables as x.

Here $\frac{dz}{dy} = 0$, and (α) becomes $Q\frac{dz}{dx} = \left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}z$, or $\frac{dz}{z} = \frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}dx$; and $lz = \int \frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}dx$, or $z = e^{\int \frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}dx}$.

If z is a sole function of y, then (α) becomes $P\frac{dz}{dy} + \left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\}z = 0$, or $\frac{dz}{z} = \frac{1}{P}\left\{\frac{dQ}{dx} - \frac{dP}{dy}\right\}dy$ and $z = e^{\int \frac{1}{P}\left\{\frac{dQ}{dx} - \frac{dP}{dy}\right\}dy}$.

When z is a sole function of x, $\frac{dz}{dx}$ and consequently $\frac{1}{Q}\left\{\frac{dP}{dy}-\frac{dQ}{dx}\right\}$ is also a sole function of x. Similarly, when z is a sole function of y, $\frac{1}{P}\left\{\frac{dQ}{dx}-\frac{dP}{dy}\right\}$ is also a sole function of y; but since the converse of this proposition does not hold good, it leads to a method of finding z which is merely tentative.

Cor. When z is a sole function of x, it is of the form $e^{\int X dx}$ unless $\int \frac{1}{Q}\left\{\frac{dP}{dy}-\frac{dQ}{dx}\right\}dx$ is also a logarithm; and when a sole function of y, it is of the form $e^{\int Y dy}$.

29. *Conversely, if the form* $Pdx + Qdy$ *is rendered an exact fluxion by the factor* $\frac{1}{Px+Qy}$, *it is homogeneous.*

Let $\frac{P}{Q}=t$; then $\frac{Pdx+Qdy}{Px+Qy}$ or $\frac{tdx}{tx+y}+\frac{dy}{tx+y}$ . . . is an exact fluxion, and consequently (art. 23),

$$\frac{\frac{dt}{dy}}{tx+y}-\frac{t\left\{x\frac{dt}{dy}+1\right\}}{(tx+y)^2}=-\frac{t+x\frac{dt}{dx}}{(tx+y)^2};\text{ or } y\frac{dt}{dy}+x\frac{dt}{dx}=0;$$

wherefore (art. 8) t or $\frac{P}{Q}=F\left(\frac{y}{x}\right)$.

30. *There is an infinite number of factors which renders a form of the first order and degree an exact fluxion.*

For let $z(Pdx + Qdy) = du$ be an exact fluxion; then, multiplying by φu, $z\varphi u(Pdx + Qdy) = \varphi u du$ which is an exact fluxion, whatever form we assign to φu.

Hence it appears that if z is a factor which renders integrable $Pdx + Qdy$ so that $z(Pdx + Qdy) = du$, then $z\varphi u$ is also a factor which possesses the same property.

Thus $\frac{1}{y^2}\times$ any function of $\frac{x}{y}$ will render $ydx - xdy$ an exact fluxion.

Cor. If there is a form $Pdx + Qdy + Rdx + Sdy$ such that there are two factors z and t which respectively

render $\dot{P}dx + \dot{Q}dy$ and $\dot{R}dx + \dot{S}dy$ exact fluxions, so that we have $z(\dot{P}dx + \dot{Q}dy) = du$ and $t(\dot{R}dx + \dot{S}dy) = dv$; then $z\dot{F}u$ and tfv comprehend all the factors which render the proposed an exact fluxion; and if they are of such form that they may be supposed equal, there will result from the equation the required factor.

31. EXAMPLES.

Ex. 1. $ydy + xdx - 2ydx = du$.

Here $\dot{P}x + \dot{Q}y = (x - 2y)x + y^2 = (y - x)^2$; and $\frac{ydy + xdx - 2ydx}{(y-x)^2} = du$ is an exact fluxion, and

$u = l\frac{y-x}{c} - \frac{y}{y-x}$. (Art. 25. Ex. 5.)

Ex. 2. $ydx - xdy = du$.

Here $\dot{P}x + \dot{Q}y = 0$, and the method of art. 28, case 2, fails; but since $\frac{d\dot{P}}{dy} - \frac{d\dot{Q}}{dx} = 2$, the equation ($\alpha$) becomes $\frac{dz}{z} = -\frac{2}{x}dx$, $\therefore z = \frac{c}{x^2}$; and $\frac{c}{x^2}(ydx - xdy)$ is an exact fluxion.

Ex. 3. $xdy - 2ydx + bdx = du = 0$.

Since $\frac{1}{\dot{Q}}\left\{\frac{d\dot{P}}{dy} - \frac{d\dot{Q}}{dx}\right\} = -\frac{3}{x}$, z may be a sole function of x, and we have $\frac{dz}{z} = -\frac{3dx}{x}$, or $z = \frac{1}{x^3}$; and du becomes $\frac{dy}{x^2} - \frac{2ydx}{x^3} + \frac{bdx}{x^3}$; which is an exact fluxion, and the required primitive is $\frac{y}{x^2} - \frac{b}{2x^2} + c = 0$, or $cx^2 + y = \frac{b}{2}$.

Ex. 4. $aydy - by^2dx + cxdx = 0 = du$.

$\dot{P} = cx - by^2$

$\dot{Q} = ay$

Here $\frac{1}{\dot{Q}}\left\{\frac{d\dot{P}}{dy} - \frac{d\dot{Q}}{dx}\right\} = -\frac{2b}{a}$ $\therefore$ . . . . .

$\frac{dz}{z} = -\frac{2bdx}{a}$ and $z = e^{-\frac{2bx}{a}}$; and du becomes $ce^{-\frac{2bx}{a}}xdx + ae^{-\frac{2bx}{a}}ydy - be^{-\frac{2bx}{a}}y^2dx$ an exact

fluxion, $\therefore \int ce^{-\frac{2bx}{a}} x dx + \frac{a}{2} e^{-\frac{2bx}{a}} y^2 = c$, or $by^2 - cx - \frac{ac}{2b} \cdots = ce^{\frac{2bx}{a}}$. (Vid. 18, ex. 8.)

Ex. 5. The form $dy + xydx$.

Here $\frac{1}{y}$ is the requisite factor, and $\frac{dy}{y} + xdx$ is an exact fluxion $= du$; $\therefore u = ly + \int xdx = l.e^{\int xdx} y$ $\therefore$ (30) the canonical form is $\frac{1}{y}\varphi.e^{\int xdx} y$.

Ex. 6. The linear equation $dy + Rydx = Sdx$ is integrable by the introduction of a factor.

For the equation is $(Ry - S)dx + dy = 0$, and consequently $\frac{1}{Q}\left\{\frac{dP}{dy} - \frac{dQ}{dx}\right\} = R$, or the requisite factor is $e^{\int Rdx}$; and multiplying the proposed by this, we have $e^{\int Rdx} dy + ye^{\int Rdx} Rdx = e^{\int Rdx} Sdx$; and integrating $e^{\int Rdx} y = \int e^{\int Rdx} Sdx + c$, and $y = e^{-\int Rdx}\{\int e^{\int Rdx} Sdx + c\}$.

Ex. 7. $aydx + bxdy + x^m y^n (cydx + exdy) = 0$.

$\frac{1}{xy}$ is a factor which will render the two first terms an exact fluxion, and the fluent is $alx + bly$ or $l.x^a y^b$; wherefore $\frac{1}{xy} F.x^a y^b$ is also a factor which will render the same an exact fluxion.

Similarly $\frac{1}{x^{m+1} y^{n+1}}$ renders the third term an exact fluxion, whose fluent is $l.x^e y^e$; and consequently $\frac{1}{x^{m+1} y^{n+1}} f.x^e y^e$ renders this term an exact fluxion; and to find a factor which renders the whole integrable, make $\frac{1}{xy} F.x^a y^b = \frac{1}{x^{m+1} y^{n+1}} f.x^e y^e$ or $F.x^a y^b = \frac{1}{x^m y^n} f.x^e y^e$.

Now, suppose $F.x^a y^b = x^{ka} y^{kb}$ and $f.x^e y^e = x^{re} y^{re}$, $\therefore$ we have $\left.\begin{array}{l} ka = rc - m \\ kb = re - n \end{array}\right\} \therefore k = \frac{nc - me}{ae - bc}$ and $r = \frac{na - mb}{ae - bc}$.

Multiply then the two first terms by $x^{ka-1}y^{kb-1}$, and the third by $x^{rc-m-1}y^{re-n-1}$, and there results $ay^{kb}x^{ka-1}dx + bx^{ka}y^{kb-1}dy + x^{rc-1}y^{re-1}(cydx + exdy) = 0$; and integrating, $\frac{1}{k}x^{ka}y^{kb} + \frac{1}{r}x^{rc}y^{re} = c$ is the required equation, in which the values of k and r are known.

Cor. 1. If either $nc - me = 0$, or $na - mb = 0$, we have $k = 0$, and a logarithm enters into the resulting equation.

Cor. 2. If $ae - bc = 0$, $k = \infty$, and this method fails; but in this case, since $e = \frac{bc}{a}$, the equation is

$$aydx + bxdy + \frac{c}{b}x^m y^n(aydx + bxdy) = 0,$$

$$\text{or } (aydx + bxdy)\left\{1 - \frac{c}{b}x^m y^n\right\} = 0.$$

This equation is satisfied either by $aydx + bxdy = 0$; which integrated, gives $x^a y^b = \text{const.}$; or by

$1 - \frac{c}{b}x^m y^n = 0$; of which the latter is a mere algebraical solution of the original equation.

Ex. 8. $axdy + 2aydx = xydy$.

$P = 2ay$

$Q = (a-y)x$

Here $\frac{1}{P}\left\{\frac{dQ}{dx} - \frac{dP}{dy}\right\} = \frac{1}{2ay}(a - y - 2a) \cdots = -\frac{a+y}{2ay}$; if then z is a sole function of y,

$$\frac{dz}{z} = -\frac{ady}{2ay} - \frac{dy}{2a} \text{ and } lz = \frac{1}{l.y^{\frac{1}{2}}} - \frac{y}{2a}.$$

The requisite factor in this case is not a sole function of y; it is $\frac{1}{xy}$; and the primitive is $x^a y = ce^{\frac{y}{a}}$.

32. *An equation may be sometimes integrated by dividing it into exact fluxions by means of factors, and substituting new variables for the fluxions.*

Ex. 1. $x^m(ydx + xdy) = y^n(ydx - xdy)$.

Here $x^m d.xy = y^{n+2} d.\frac{x}{y}$.

Let $v = xy$, $w = \frac{x}{y}$ $\Big\}$ $\therefore x = \sqrt{vw}$ and $y = \sqrt{\frac{v}{w}}$; and by substitution, $v^{\frac{m}{2}} w^{\frac{m}{2}} dv = \frac{v^{\frac{n+2}{2}} dw}{w^{\frac{n+2}{2}}}$ $\therefore v^{\frac{m-n-2}{2}} dv = w^{-\frac{m+n+2}{2}} dw$ $\therefore$ integrating, $\frac{2v^{\frac{m-n}{2}}}{m-n} + \frac{2w^{-\frac{m+n}{2}}}{m+n} = 2c$, or . . . . . .

$$\frac{x^{\frac{m-n}{2}} y^{\frac{m-n}{2}}}{m-n} + \frac{x^{-\frac{m+n}{2}} y^{\frac{m+n}{2}}}{m+n} = c.$$

Ex. 2. $a(xdy - ydx) = (xdx + ydy)\sqrt{x^2 + y^2 - a^2}$.

Here $ax^2 d.\frac{y}{x} = \frac{1}{3} d.(x^2 + y^2 - a^2)^{\frac{3}{2}}$.

Let $\frac{y}{x} = v$, $x^2 + y^2 - a^2 = w$ $\Big\}$ $\therefore x^2 + v^2 x^2 - a^2 = w$ $\therefore x^2 = \frac{w + a^2}{1 + v^2}$;

and by substitution $\frac{a(w + a^2)}{1 + v^2} dv = \frac{1}{3} d.w^{\frac{3}{2}} = \frac{1}{2} w^{\frac{1}{2}} dw$ or $\frac{adv}{1 + v^2}$

$= \frac{\frac{1}{2} w^{\frac{1}{2}} dw}{w + a^2}$ $\therefore a\text{tan.}^{-1} v = w^{\frac{1}{2}} - a\text{tan.}^{-1} \frac{w^{\frac{1}{2}}}{a}$, or $a\text{tan.}^{-1} \frac{y}{x}$

$= \sqrt{x^2 + y^2 - a^2} - a\,\text{tan.}^{-1} \frac{\sqrt{x^2 + y^2 - a^2}}{a}$, or . . . .

$$\frac{ay + x\sqrt{x^2 + y^2 - a^2}}{ax - y\sqrt{x^2 + y^2 - a^2}} = \text{tan.} \frac{\sqrt{x^2 + y^2 - a^2}}{a}.$$

Ex. 3. $ydy - xdx - \frac{bdy}{x^5} + \frac{bydx}{x^6} = 0$.

Here $\frac{1}{2} d.(y^2 - x^2) = \frac{b}{x^4} d.\frac{y}{x}$.

Let $v = y^2 - x^2$, $w = \frac{y}{x}$ $\Big\}$ $\therefore v = x^2 w^2 - x^2$ or $x^4 = \frac{v^2}{(w^2 - 1)^2}$ $\therefore$

by substitution, $dv = \frac{2b(w^2-1)^2 dw}{v^2}$, or $v^2 dv = 2b(w^2-1)^2 dw$

$\therefore v^3 = 6b\left\{\frac{w^5}{5} - \frac{2w^3}{3} + w\right\}$, or . . . . . .

$(y^2 - x^2)^3 = 6b\left\{\frac{y^5}{5x^5} - \frac{2y^3}{3x^3} + \frac{y}{x}\right\}$.

Ex. 4. $\frac{adx}{x} + \frac{bdy}{y} = \frac{x^m dx}{ky^n}$.

Here $y^n dl \,.\, x^a y^b = \frac{1}{k} x^m dx$.

Let $x^a y^b = v \therefore y^n = v^{\frac{n}{b}} x^{-\frac{na}{b}}$; and by substitution,

$v^{\frac{n}{b}} x^{-\frac{na}{b}} \frac{dv}{v} = \frac{1}{k} x^m dx$; or $v^{\frac{n}{b}-1} dv = \frac{1}{k} x^{m+\frac{na}{b}} dx$; and integrating, $\frac{bk}{n} v^{\frac{n}{b}}$, or $\frac{bk}{n} x^{\frac{na}{b}} y^n = \frac{x^{m+\frac{na}{b}+1}}{m + \frac{na}{b} + 1} + c$.

Ex. 5. $aydx + bxdy + x^m y^n (cydx + exdy) = 0$.

Here $dl \,.\, x^a y^b + x^m y^n dl \,.\, x^c y^e = 0$.

Let $\left.\begin{matrix} x^a y^b = v \\ x^c y^e = w \end{matrix}\right\} \therefore \frac{v^{\frac{1}{a}}}{y^{\frac{b}{a}}} = \frac{w^{\frac{1}{c}}}{y^{\frac{e}{c}}} \therefore y^{\frac{ae-bc}{ac}} = \frac{w^{\frac{1}{c}}}{v^{\frac{1}{a}}} \therefore$ . . . .

$y = v^{\frac{-c}{ae-bc}} \times w^{\frac{a}{ae-bc}}$.

Similarly $x = v^{\frac{e}{ae-bc}} \times w^{\frac{-b}{ae-bc}} \therefore$ by substitution,

$\frac{dv}{v} + v^{\frac{me-nc}{ae-bc}} \times w^{\frac{na-mb}{ae-bc}} \frac{dw}{w} = 0$, or $v^{\frac{nc-me}{ae-bc}-1} dv + w^{\frac{na-mb}{ae-bc}-1} dw = 0$

$\therefore$, integrating, $\frac{v^{\frac{nc-me}{ae-bc}}}{nc-me} + \frac{w^{\frac{na-mb}{ae-bc}}}{na-mb} = c$, in which v and w are to be replaced by their values $x^a y^b$ and $x^c y^e$. (Vid. art. 31, ex. 7.)

Cor. If either $nc - me = 0$, or $na - mb = 0$, the fluents of the final equation become lv or lw.

This very useful method of integration is nearly allied to the one given in art. 30, cor. The proposed should be di-

vided into such exact fluxions that the new equation may be under a form the most convenient for integration.

Ex. 6. $xdx + aydx + 2xydy - 2a^2ydy = 0$.

If we divide this into $xd(x + y^2) + ayd(x - 2ay) = 0$, the equation between v and w will be as difficult to integrate as the original. But, since $(x+ay)dx+(x-a^2)2ydy=0$,

$$\therefore 0 = \frac{x+ay}{x-a^2}dx + 2ydy = dx + \frac{a(a+y)dx}{x-a^2} + 2ydy \cdots$$

$$= dx + (a + y)\left\{\frac{adx}{x-a^2} + \frac{2ydy}{a+y}\right\} \cdots$$

$$= dx + (a + y)\left\{\frac{adx}{x-a^2} + 2dy - \frac{2ady}{a+y}\right\} \cdots$$

$$= dx + 2(a + y)dy + a(a + y)dl.\frac{x-a^2}{(a+y)^2}.$$

Let $v = x + 2ay + y^2$, $w = \dfrac{x-a^2}{(a+y)^2}$ $\Big\}$ $\therefore v - (a + y)^2w = (y + a)^2$,

or $a + y = \sqrt{\dfrac{v}{w+1}}$; and by substitution,

$$0 = dv + a\sqrt{\frac{v}{w+1}}\,\frac{dw}{w}, \text{ or } \frac{dv}{v^{\frac{1}{2}}} + \frac{adw}{w\sqrt{w+1}} = 0 \therefore \text{ inte-}$$

grating, $2v^{\frac{1}{2}} + al.\dfrac{(\sqrt{w+1}-1)^2}{w} = 2c$, or

$$\sqrt{x + 2ay + y^2} + al.\frac{\sqrt{x+2ay+y^2}-(a+y)}{\sqrt{x-a^2}} = c.$$

Since the original equation in the process of integration has been divided by $x - a^2$ and also by $\sqrt{v}$, the requisite factor is $\dfrac{1}{(x-a^2)\sqrt{x^2+2ay+y^2}}$.

33. *Required the functions* A *and* B *so that an equation of the form* $Aydx + Bdy + ydy = 0$ *may be rendered integrable per se by a factor of the given form* $\dfrac{1}{y^3+Cy^2+Dy}$; *where* A, B, C, D *are all functions of* x.

Here $\mathrm{P} = \mathrm{A}y$, $\mathrm{Q} = \mathrm{B} + y$ | whence the equation (α), art. 28, becomes

$$-\mathrm{A}y\frac{3y^2+2\mathrm{C}y+\mathrm{D}}{(y^3+\mathrm{C}y^2+\mathrm{D}y)^2} \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+(\mathrm{B}+y)\frac{\mathrm{C}'y^2+\mathrm{D}'y}{(y^3+\mathrm{C}y^2+\mathrm{D}y)^2}+\frac{\mathrm{A}-\mathrm{B}'}{y^3+\mathrm{C}y^2+\mathrm{D}y}=0;$$ wherefore transposing and dividing by y,

$(\mathrm{B}+y)(\mathrm{C}'y+\mathrm{D}')-\mathrm{B}'(y^2+\mathrm{C}y+\mathrm{D})=2\mathrm{A}y^2+\mathrm{A}\mathrm{C}y$, which obtains whatever be the value of y; whence $\mathrm{B}\mathrm{D}'-\mathrm{B}'\mathrm{D}=0$; or integrating, $\mathrm{D}=a\mathrm{B}$ where a is an arbitrary constant.

Also, dividing by y, $(\mathrm{B}+y)\mathrm{C}'+\mathrm{D}'-\mathrm{B}'(y+\mathrm{C})=2\mathrm{A}y+\mathrm{A}\mathrm{C}$, or $\mathrm{B}\mathrm{C}'+\mathrm{D}'-\mathrm{B}'\mathrm{C}=\mathrm{A}\mathrm{C}$, and $\mathrm{C}'-\mathrm{B}'=2\mathrm{A}$; whence by elimination, $\mathrm{B}\mathrm{C}'+(a-\mathrm{C})\mathrm{B}'=\frac{\mathrm{C}(\mathrm{C}'-\mathrm{B}')}{2}$, or

$2\mathrm{B}\mathrm{C}'+(2a-\mathrm{C})\mathrm{B}'=\mathrm{C}\mathrm{C}'$, or $d\mathrm{B}+\frac{2\mathrm{B}d\mathrm{C}}{2a-\mathrm{C}}=\frac{\mathrm{C}d\mathrm{C}}{2a-\mathrm{C}}$ a linear equation; and consequently $\mathrm{B}=b(2a-\mathrm{C})^2+\mathrm{C}-a$ (4, 18, ex. 4); and A, which $=\frac{\mathrm{C}'-\mathrm{B}'}{2}=b(2a-\mathrm{C})\mathrm{C}'$; where C may be any function whatever of x, but it is necessary that $\mathrm{D}=a\mathrm{B}$.

34. *Required to find the form of* X, X′ *functions of* x, *so that* $\mathrm{X}y dx+\mathrm{X}'dx+y dy$ *shall be rendered integrable per se by a function of the form* $\frac{1}{(y+fx)^n}$.

$z=\frac{1}{(y+fx)^n}$, and equation (α), art. 28, becomes

$-n(\mathrm{X}y+\mathrm{X}')+y.nf'x+\mathrm{X}(y+fx)=0$, or $(n-1)\mathrm{X}y-nf'xy+n\mathrm{X}'-\mathrm{X}fx=0$, which obtains whatever be the value of y; wherefore $(n-1)\mathrm{X}-nf'x=0$ and $n\mathrm{X}'-\mathrm{X}fx=0$ are the equations from which the values of X and of X′ may be determined.

Cor. Conversely, if X and X′ are given, fx may be found from either of the above equations.

Ex. $\frac{mxydx}{\sqrt{1+x^2}}-xdx+(m+1)ydy=0$, or

$$\frac{m}{m+1}\frac{xydx}{\sqrt{1+x^2}}-\frac{xdx}{m+1}+ydy=0.$$

Here $x = \frac{mx}{(m+1)\sqrt{1+x^2}}$ and $x' = -\frac{x}{m+1}$; wherefore from the second equation, we have

$-\frac{nx}{m+1} - \frac{mx}{(m+1)\sqrt{1+x^2}} fx = 0$, or $n = -m$ and $fx = \sqrt{1+x^2}$ $\therefore z = (y + \sqrt{1+x^2})^m$.

If m be assumed any integer, 2 for instance, and the binomial be expanded and multiplied into the proposed equation, the result will prove to be an exact fluxion.

35. It has been shown at the beginning of this chapter, art. 2, that the algebraick relation of transcendentals may be sometimes found. There are other more complicated forms in which the variables are separated, which admit of algebraick fluents, though the fluent of each part is a transcendental.

Ex. 1. $\frac{dx}{\sqrt{a+bx+ex^2}} - \frac{dy}{\sqrt{a+by+ey^2}} = 0.$

Suppose $\frac{dx}{\sqrt{a+bx+ex^2}} = dt = \frac{dy}{\sqrt{a+by+ey^2}}$; then $\frac{dx^2}{dt^2} = a + bx + ex^2$ and $\frac{dy^2}{dt^2} = a + by + ey^2$.

Differentiating on the hypothesis that t is the principal variable, $\frac{2d^2x}{dt^2} = b + 2ex$ and $\frac{2d^2y}{dt^2} = b + 2ey$. Adding these and substituting $x + y = p$, we have $\frac{d^2p}{dt^2} = b + ep$; and to integrate this equation of the second order, multiply by $2dp$, and it becomes $\frac{2dpd^2p}{dt^2} = 2bdp + 2epdp$; of which the complete fluent is $\frac{dp^2}{dt^2} = c + 2bp + ep^2$, or $\frac{dp}{dt} = \sqrt{c + 2bp + ep^2}$. But $\frac{dp}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \sqrt{a + bx + ex^2} + \sqrt{a + by + ey^2}$; wherefore the required fluent is $\sqrt{a + bx + ex^2} + \sqrt{a + by + ey^2} = \sqrt{c + 2bp + ep^2} = \sqrt{c + 2b(x + y) + e(x + y)^2}$.

Similarly it may be shown that the fluent of

$$\frac{dx}{\sqrt{a+bx+ex^2}}+\frac{dy}{\sqrt{a+by+ey^2}}=0 \text{ is}$$

$$\sqrt{a+bx+ex^2}-\sqrt{a+by+ey^2}=\sqrt{c+2b(x+y)+e(x+y)^2}.$$

Thus, if $\frac{dx}{\alpha+\beta x}+\frac{dy}{\alpha+\beta y}=0$; considering $\alpha+\beta x$ and $\alpha+\beta y$ as the square roots of $a+bx+ex^2$ and $a+by+ey^2$, we have $a=\alpha^2$, $b=2\alpha\beta$ and $e=\beta^2$, and consequently the fluent is $c=\alpha(x+y)+\beta xy$; which is the same as that which would result from considering the proposed as the fluxion of the sum of two logarithms.

Ex. 2. $\frac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}}-\frac{dy}{\sqrt{a+by+cy^2+ey^3+fy^4}}=0.$

This by substitution is $\frac{dx}{\sqrt{\text{X}}}-\frac{dy}{\sqrt{\text{Y}}}=0$, where

$\text{X}=a+bx+cx^2+cx^3+fx^4$ and $\text{Y}=a+by+cy^2+ey^3+fy^4$.

Suppose, as before, $\frac{dx}{\sqrt{\text{X}}}=dt=\frac{dy}{\sqrt{\text{Y}}}$; then $\frac{dx^2}{dt^2}=\text{X}$ and $\frac{dy^2}{dt^2}=\text{Y}$.

Substitute $x+y=p$, $x-y=q$.

Now $\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}$, which $=\frac{dx+dy}{dt}\cdot\frac{dx-dy}{dt}=\frac{dpdq}{dt^2}$, $\ldots$

$=\text{X}-\text{Y}=b(x-y)+c(x^2-y^2)+e(x^3-y^3)+f(x^4-y^4)$

$=bq+cpq+\frac{eq}{4}(3p^2+q^2)+\frac{fpq}{2}(p^2+q^2)$, or

$$\frac{dpdq}{qdt^2}=b+cp+\frac{e}{4}(3p^2+q^2)+\frac{fp}{2}(p^2+q^2)\ (\alpha).$$

Differentiating on the hypothesis that t is the principal variable, we have

$$\left.\begin{aligned}\frac{2d^2x}{dt^2}=\frac{d\text{X}}{dx}&=b+2cx+3ex^2+4fx^3\\ \frac{2d^2y}{dt^2}=\frac{d\text{Y}}{dy}&=b+2cy+3ey^2+4fy^3\end{aligned}\right\}\text{; wherefore, adding these,}$$

$\frac{d^2p}{dt^2} = b + cp + \frac{3e}{4}(p^2 + q^2) + \frac{fp}{2}(p^2 + 3q^2)$; from which subtracting (α) there results

$\frac{d^2p}{dt^2} - \frac{dpdq}{qdt^2} = \frac{eq^2}{2} + fpq^2$ or $\frac{1}{dt^2}\left\{\frac{2dpd^2p}{q^2} - \frac{2dp^2dq}{q^3}\right\}$ . . .

$= cdp + 2fpdp$; whose fluent is $\frac{dp^2}{q^2dt^2} = c + ep + fp^2$, or

$\frac{dp}{dt}$, which $= \frac{dx}{dt} + \frac{dy}{dt}$, $= q\sqrt{c + ep + fp^2}$; wherefore the required fluent is $\sqrt{a + bx + cx^2 + ex^3 + fy^4} + \ldots$

$\sqrt{a+by+cy^2+ey^3+fy^4} = (x-y)\sqrt{c+e(x+y)+f(x+y)^2}$.

Similarly it may be shown that the fluent of

$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$ is . . . . . .

$\sqrt{X} - \sqrt{Y} = (x - y)\sqrt{c + e(x + y) + f(x + y)^2}$.

Since $\sqrt{X} - \sqrt{Y} = \frac{X - Y}{\sqrt{X} + \sqrt{Y}}$, this last may be under the form $\sqrt{X} - \sqrt{Y}$ . . . .

$$= \frac{b+c(x+y)+e(x^2+xy+y^2)+f(x^3+x^2y+xy^2+y^3)}{\sqrt{c+e(x+y)+f(x+y)^2}}.$$

35. MISCELLANEOUS PRAXIS.

1. $\frac{dx}{\sqrt{1+x^2}} + \frac{dy}{\sqrt{1+y^2}} = 0 \therefore$ . . . . . $(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = c^2$.
2. $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0 \therefore y = \frac{c-x}{1+cx}$.
3. $mydx + nxdy = 0 \therefore x^m y^n = c$.
4. $y^2dx + x^2dy = 0 \therefore xy = c(x + y)$.
5. $dx\sqrt{1 + y^2} - xdy = 0 \therefore cx = y + \sqrt{1 + y^2}$.
6. $\sqrt{1 + y^2}(xdy + ydx) = ydy \therefore xy = \sqrt{1 + y^2} + c$.
7. $xdy - ydx - (x^2 + 1)dx = 0 \therefore y = x^2 + cx - 1$.
8. $ydx - (x + y)dy = 0 \therefore y = ce^{\frac{x}{y}}$.
9. $y^2dx + (xy + x^2)dy = 0 \therefore xy^2 = c^2(x + 2y)$.

10. $x^2dy - (x^2 - ay^2)dx = 0 \therefore x^n = \dfrac{c(y-mx)}{y+nx}$ where m and n are the roots of $z^2 + \dfrac{z}{a} - \dfrac{1}{a} = 0$.

11. $\dfrac{x^2+xy}{x-y}dy = ydx \therefore xy = ce^{\frac{x}{y}}$.

12. $dx + \dfrac{ydy}{x+\sqrt{x^2+y^2}} = 0 \therefore c = x + \sqrt{x^2+y^2}$.

13. $\dfrac{dy}{dx} = ax + by + c \therefore ce^{bx} = b(ax + by + c) + a$.

14. $\dfrac{dy}{dx} = \dfrac{10+6y-4x}{6x-9y+3} \therefore \dfrac{2x+3y}{3} = l(2x - 3y - 2)$.

15. $\dfrac{dy}{dx} = \dfrac{2x-y}{a+4y-x} \therefore 2vdv - wdv + vdw - 4wdw = 0$, where $v = x - \dfrac{a}{7}$, $w = y - \dfrac{2a}{7}$.

16. $xdx + dy = ay^{\frac{1}{2}}dx$
17. $dy = dx\sqrt{x^2 + ay}$
} become homogeneous by $y = z^2$.

18. $dy + \dfrac{ay+be^x}{m+ye^{-x}}dx = 0$ becomes homogeneous by $e^x = z$.

19. $3x^2dy - 3axdy = aydx \therefore xy^3 = c(x - a)$.

20. $ady = ydx + bx^2dx \therefore y = ce^{\frac{x}{a}} - b(x^2 + 2ax + 2a^2)$.

21. $dy + ydx = (1 + x)dx \therefore y = x + ce^{-x}$.

22. $dy + ydx = (1 + x^3)dx \therefore y = x^3 - 3x^2 + 6x - 5 + ce^{-x}$.

23. $dy - x^2dy + xydx = adx \therefore$ . . . . . .
$y = ax + c\sqrt{1 - x^2}$.

24. $dy + \dfrac{(n-1)ydx}{x} = \dfrac{(n-1)dx}{x} + \dfrac{mdx}{1-x} + \dfrac{xdx}{(1-x)^2} \therefore$
$y = 1 + \dfrac{1}{x^{n-1}}\left\{m\int\dfrac{x^{n-1}dx}{1-x} + \int\dfrac{x^ndx}{(1-x)^2} + c\right\}$.

25. $ay^mdy = nxdy - ydx \therefore x = cy^n - \dfrac{ay^m}{m-n}$.

26. $ydy - \dfrac{y^2dx}{\sqrt{a^2+x^2}} = a^2dx \therefore \dfrac{y^2}{2} = \dfrac{cz^3 - 3a^2z^2 - a^4}{3z}$ . . .
where $z = x + \sqrt{a^2 + x^2}$.

27. $yxdy - a^2dx + bydy = 0 \therefore x + b = ce^{\frac{y}{2a^2}}$.

28. $axdy + 2aydx = xydy \therefore yx^2 = ce^{\frac{y}{a}}$.

29. $\frac{dx}{dy} = xy + x^2y^3 \therefore \frac{1}{x} = ce^{-\frac{y^2}{2}} - y^2 + 2$.

30. $ady = by\frac{dx}{x} + y^2dx \therefore \frac{1}{y} = cx^{-\frac{b}{a}} - \frac{x}{a+b}$.

31. $(a^2y + x^3)dx + (b^3 + a^2x)dy = 0 \therefore$ . . . . . .

$a^2xy + \frac{x^4}{4} + b^3y = c^4$.

32. $dy + \frac{xydx}{1-x^2} = xy^{\frac{1}{2}}dx \therefore y^{\frac{1}{2}} = c(1 - x^2)^{\frac{1}{4}} - \frac{1-x^2}{3}$.

33. $dy + y^{-1}dx = dx \therefore y = e^{x-y}$.

34. $\frac{dy}{dx} = a\sin.x + by$ . . . . . .

$\therefore y = ce^{bx} - \frac{a}{1+b^2}(b\sin.x + \cos.x)$.

35. $\frac{dy}{dx} + \frac{2y}{x} - \frac{x^2}{y} = \frac{x^3}{y} \therefore y^2 = cx^{-4} + \frac{x^4}{4} + \frac{2x^3}{7}$.

36. $du = \frac{xydy - y^2dx}{x^2\sqrt{x^2-y^2}} \therefore u = \frac{\sqrt{x^2+y^2}}{x}$.

37. $xdx + ydy - ydx + xdy = 0 = du$ . . . . . .

$\therefore z = \frac{1}{x^2+y^2}$ and $c = l\sqrt{x^2+y^2} + \tan.^{-1}\frac{y}{x}$.

38. $(x - 3y)dx - (y - 3x)dy = 0$ . . . . . .

$\therefore z = \frac{1}{x^2-y^2}$ and $(x + y)^2 = c(x - y)$.

39. $(2a - y)dx + (2x - y)dy = 0$ . . . . . .

$\therefore z = \frac{1}{(2a-y)^3}$ and $c = \frac{x}{(2a-y)^2} - \int\frac{ydy}{(2a-y)^3}\cdots$

$= \frac{x-y+a}{(2a-y)^2}$.

40. $ay^2dx - axydy = 2xdx \therefore z = \frac{1}{x^3}$ and $ay^2 = cx^2 + 4x$.

41. $\dfrac{ydx}{x^2} - \dfrac{2dy}{x} + by^n dy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$

$\therefore z = y$ and $\dfrac{by^{n+2}}{n+2} - \dfrac{y^2}{x} = c.$

42. $x^3dy + \left\{ 4x^2y - (1 - x^2)^{-\frac{1}{2}} \right\} dx = 0 \quad . \quad . \quad . \quad .$

$\therefore z = x$ and $x^4y + \sqrt{1 - x^2} = c.$

43. $\dfrac{2x^2dx + xydy + y^2dx}{x^4 + x^2y^2 + a^4} = \dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} \quad . \quad . \quad . \quad . \quad . \quad .$

$\therefore l\sqrt{x^2 + y^2} = \dfrac{1}{a^2} \tan.^{-1} \dfrac{x\sqrt{x^2 + y^2}}{a^2}.$

44. $\dfrac{d\theta}{\sqrt{1 - e^2 \sin.^2\theta}} \pm \dfrac{d\phi}{\sqrt{1 - e^2 \sin.^2\phi}} = 0 \quad . \quad . \quad . \quad . \quad .$

$\therefore \sqrt{1 - e^2 \sin.^2\theta} \pm \sqrt{1 - e^2 \sin.^2\phi} = c \sin.(\theta \pm \phi).$

CHAPTER V.

Fluxional Equations of a higher degree than the first.

1. When a fluxional equation is of a higher degree than the first, we know that it is not an immediate derivative; but that it has been deduced by the elimination of a constant, which is of the same degree as the equation.

The canonical form, free from radicals, is $dy^n + \mathrm{v}dy^{n-1}dx + \mathrm{q}dy^{n-2}dx^2 \cdots + \mathrm{v}dx^n = 0$, which divided by dx^n, becomes $\left(\frac{dy}{dx}\right)^n + \mathrm{p}\left(\frac{dy}{dx}\right)^{n-1} \cdots + \mathrm{t}\frac{dy}{dx} + \mathrm{v} = 0$.

This is an equation of n dimensions, which is to be resolved into its simple factors by the Theory of Equations (Alg. Part 2). We here suppose that all the roots of the equation are real; if it contain imaginary roots, some of the factors will be trinomials, which are to be resolved into quadratick divisors, as in ch. 1, art. 6.

Let the roots be p', p'', p''', $\cdots$ $p^{(n)}$; then the factors, viz. $\frac{dy}{dx} - p' = 0$, $\frac{dy}{dx} - p'' = 0$, $\cdots$, being equations of the first degree, may be integrated as in the preceding chapter. Any one of them will satisfy the proposed equation, as will also the continued product of any number of them.

Integrate then the n equations $\frac{dy}{dx} - p' = 0$, $\frac{dy}{dx} - p'' = 0$, $\cdots$ introducing the same constant a, and there will result the required primitive.

The resulting equation is obtained under a resolved form, but by involution it may be cleared of radicals.

Cor. The values of the constant in terms of (x, y), as deduced from the n simple equations, are the same as the n roots of the original primitive solved with respect to a.

Ex. 1. $dy^2 - b^2dx^2 = 0$, or $p^2 - b^2 = 0$.

Here $p = \pm b$, $\therefore$ integrating and introducing the same

constant, $y = \pm bx + a$; and the required primitive is $(y - a)^2 = b^2x^2$.

Ex. 2. $ydy^2 + 2xdydx = ydx^2$, or $yp^2 + 2xp = y$.

Here $p = -\frac{x \pm \sqrt{x^2+y^2}}{y} \therefore \frac{yp+x}{\pm\sqrt{x^2+y^2}} = 1$; and integrating, $\pm\sqrt{x^2+y^2} = x + a$; whence the original primitive is $y^2 = 2ax + a^2$.

Ex. 3. $xdx + ady = b\sqrt{dx^2 + dy^2}$.

Here $x^2 + 2axp + a^2p^2 = b^2(1 + p^2)$, or . . . . . $(a^2 - b^2)p^2 + 2axp = b^2 - x^2 \therefore$ . . . . . $p = -\frac{ax}{a^2-b^2} \pm \frac{b\sqrt{x^2+a^2-b^2}}{a^2-b^2} =$, by substitution, $-\frac{ax}{r^2} \pm \frac{b\sqrt{x^2+r^2}}{r^2}$; whence

$$y = -\frac{ax^2}{2r^2} \pm \frac{b}{r^2}\left\{\tfrac{1}{2}x\sqrt{x^2+r^2} + \frac{r^2}{2}\,l\,\frac{x+\sqrt{x^2+r^2}}{c}\right\};$$

which is the required equation when resolved.

Ex. 4. $x^2dx^2 - xydydx = a^2dy^2$.

Here it will be more convenient to solve the equation with respect to x; and we have $x = \frac{yp}{2} \pm \sqrt{a^2 + \frac{y^2}{4}p} \therefore$

$p = \frac{2x}{y \pm \sqrt{4a^2+y^2}}$; and the equation is

$x^2 - \frac{y^2}{2} = \pm\int\sqrt{4a^2+y^2}dy$, which is a known transcendental form.

There are cases in which the resolution of the proposed equation may be avoided by adopting certain artifices of calculation, as in the following articles.

2. *An equation which contains only one of the variables, and which can be resolved with respect to that variable, is integrable.*

For let the equation contain only x, and let it be reduced to the form $x = \text{F}p$; then since $dy = pdx$, integrating by parts, $y = px - \int xdp = p\text{F}p - \int \text{F}p.dp$. Now $\int \text{F}p.dp$ may be integrated by the common rules, and this will give an equation between y and p; from which, p being eliminated by means of the proposed equation $x = \text{F}p$, we shall have the required primitive.

If p can be more conveniently expressed in terms of x,

we have $p = \mathrm{x}$, or $dy = \mathrm{x}dx$; and the required primitive is $y = \int \mathrm{x}dx$.

If x and p cannot be expressed in terms either of the other, it may happen that we can express each in terms of a new function v. Let $x = \mathrm{v}$, $p = \mathrm{w}$, where v and w are functions of v; then dy, which $= pdx$, $= \mathrm{w}d\mathrm{v}$ and $y = \int \mathrm{w}d\mathrm{v}$; from which v is to be eliminated by means of $x = \mathrm{v}$.

Similarly, let the equation contain only y, and let it be reduced to $y = \mathrm{F}p$; then since $dx = \frac{dy}{p}$, we have $x = \frac{y}{p} + \int \frac{ydp}{p^2} = \frac{y}{p} + \int \frac{\mathrm{F}p.dp}{p^2}$; from which the primitive may be obtained as before by eliminating p.

Ex. 1. $x(dx^2 + dy^2) = dx^2$.

Here $x = \frac{1}{1+p^2} \therefore y = px - \int \frac{dp}{1+p^2} = px - \tan.^{-1}p + a$; but from the proposed $p = \sqrt{\frac{1-x}{x}} \therefore$ the required equation is $y = \sqrt{x - x^2} - \tan.^{-1}\sqrt{\frac{1-x}{x}} + a$.

Ex. 2. $ydy^2 - bdydx = adx^2$, or $yp^2 - bp = a$.

$dx = \frac{dy}{p} \therefore x = \frac{y}{p} + \int \frac{ydp}{p^2} = \frac{y}{p} + \int \frac{a+bp}{p^4}dp = \&c.$

This example may be also integrated by the solution of a quadratick.

Ex. 3. $adxdy^3 = y(dx^2 + dy^2)^2$, or $ap^3 = y(1 + p^2)^2$.

Here $x = \frac{y}{p} + a\int \frac{pdp}{(1+p^2)^2} = \frac{y}{p} - \frac{a}{2}\frac{1}{1+p^2} + \mathrm{c}$; from which p must be eliminated by means of the proposed.

Ex. 4. $x^3dx^2 + dy^3 = axdx^2dy$, or $x^3 + p^3 = axp$.

Here the equation cannot be conveniently solved with respect to either x or p. Substitute $v = \frac{p}{x}$, or $p = vx$; and dividing by x^2, there results $x + v^3x = av$, or $x = \frac{av}{1+v^3}$

$\therefore p = \frac{av^2}{1+v^3} \therefore dy$, which $= pdx$,

$= \frac{a^2(1-2v^3)v^2dv}{(1+v^3)^3} = \frac{2a^2(\frac{1}{2}-v^3)v^2dv}{(1+v^3)^3}$ . . . . . .

$$= \frac{2a^2\{\frac{3}{2}-(1+v^3)\}v^2dv}{(1+v^3)^3} = \frac{3a^2v^2dv}{(1+v^3)^3} - \frac{2a^2v^2dv}{(1+v^3)^2} \therefore \ldots$$

$y = \frac{a^2+4v^3}{6(1+v^3)^2} + c$; from which v is to be eliminated by means of $x = \frac{av}{1+v^3}$.

3. *An equation which is homogeneous with respect to both the variables is integrable.*

For substitute $y = xz$; then dividing each term by x^n, the equation being of n dimensions, there results $z = \text{F}p$. But since $dy = xdz + zdx$ and also $= pdx$; we have $pdx = xdz + zdx$; or $\frac{dx}{x} = \frac{dz}{p-z}$ and, integrating,

$$lx = \int \frac{dz}{p-z} = -l(p-z) + \int \frac{dp}{p-z}.$$

Now, substituting $z = \text{F}p$ in this equation, we have $x = fp + c$; from which, eliminating p by means of $y = xz = x\text{F}p$, there will result the required primitive.

If p can be more conveniently expressed in terms of z, let $p = \text{z}$; then $lx = \int \frac{dz}{\text{z}-z}$; from which z is to be eliminated by means of $y = xz$.

Ex. 1. $ydx - xdy = x\sqrt{dx^2+dy^2}$, or $y - xp = x\sqrt{1+p^2}$.

Substitute $y = xz$; then, dividing by x, $z - p = \sqrt{1+p^2}$, or $z = p + \sqrt{1+p^2}$.

Also $pdx = dy = zdx + xdz \therefore \frac{dx}{x} = \frac{dz}{p-z} = \frac{-dz}{\sqrt{1+p^2}}$

$$= \frac{-dp}{\sqrt{1+p^2}} - \frac{pdp}{1+p^2} \therefore \frac{a}{x} = \sqrt{1+p^2}(p + \sqrt{1+p^2})$$

and $x = \frac{a}{\sqrt{1+p^2}(p+\sqrt{1+p^2})}$.

To eliminate p, we have $y = xz = \frac{a}{\sqrt{1+p^2}} \therefore \ldots$

$1 + p^2 = \frac{a^2}{y^2}$ and $p = \frac{\sqrt{a^2-y^2}}{y} \therefore$ by substitution,

$$x = \frac{y}{\frac{\sqrt{a^2-y^2}}{y} + \frac{a}{y}} = \frac{y^2}{a+\sqrt{a^2-y^2}}, \text{ or}$$

$x\sqrt{a^2-y^2}=y^2-ax$, and the required equation is $0=y^4-2axy^2+x^2y^2$, or $y^2+x^2=2ax$; where a is of one dimension, because the proposed equation, when cleared of radicals, is of the first degree.

Ex. 2. Required the curve whose length is a mean proportional between the ordinate and twice the abscissa.

$s=\sqrt{2xy}$, or the fluxional equation of the required curve is $\sqrt{dx^2+dy^2}=\dfrac{xdy+ydx}{\sqrt{2xy}}$ or $\sqrt{1+p^2}\sqrt{2xy}=xp+y$.

Substitute $y=xz$; then $\sqrt{1+p^2}\sqrt{2z}=p+z$. Here z may be found in terms of p by the solution of a quadratick and the equation integrated; but it will be more convenient to find p in terms of z; for which purpose we have

$$p^2-\frac{2z}{2z-1}p=\frac{z^2-2z}{2z-1} \text{ and } p=\frac{z+(1-z)\sqrt{2z}}{2z-1};\ \therefore p-z$$
$$=\frac{2z-2z^2+(1-z)\sqrt{2z}}{2z-1}=\frac{(2z+\sqrt{2z})(1-z)}{2z-1}=\frac{(1-z)\sqrt{2z}}{\sqrt{2z}-1};$$
$$\therefore \frac{dx}{x}, \text{ which } =\frac{dz}{p-z},=\frac{(\sqrt{2z}-1)dz}{(1-z)\sqrt{2z}}=\frac{dz}{1-z}-\frac{dz}{(1-z)\sqrt{2z}}.$$

To integrate $\dfrac{dz}{(1-z)\sqrt{2z}}=du$, substitute $z=v^2$; then

$$du=\frac{\sqrt{2}dv}{1-v^2}\ \therefore u=\frac{1}{\sqrt{2}}l\frac{1+v}{1-v}=\frac{1}{\sqrt{2}}l\frac{1+\sqrt{z}}{1-\sqrt{z}}\ \therefore \ldots$$

$lx=la-l(1-z)-\dfrac{1}{\sqrt{2}}l\dfrac{1+\sqrt{z}}{1-\sqrt{z}}$, from which there results $x-y=a\left\{\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right\}^{\frac{1}{\sqrt{2}}}$. (Euler, Calc. Int. vol. i. p. 450.)

Ex. 3. $\dfrac{xdy-ydx}{\sqrt{dx^2+dy^2}}=\text{F}\sqrt{x^2+y^2}$.

Substitute $x=vw$ and $w=\sqrt{x^2+y^2}$; $\therefore y=\sqrt{w^2-x^2}$
$=w\sqrt{1-v^2}\ \therefore dy=-\dfrac{vwdv}{\sqrt{1-v^2}}+\sqrt{1-v^2}dw$ and
$dx=wdv+vdw$; $\therefore$ . . . . . .

$xdy - ydx = -\frac{v^2w^2dv}{\sqrt{1-v^2}} - w^2\sqrt{1-v^2}dv = -\frac{w^2dv}{\sqrt{1-v^2}}$.

Also, $dx^2 + dy^2 = w^2dv^2 + v^2dw^2 + \frac{v^2w^2dv^2}{1-v^2} + (1-v^2)dw^2$

$= w^2dv^2 + \frac{v^2w^2dv^2}{1-v^2} + dw^2 = \frac{w^2dv^2 + (1-v^2)dw^2}{1-v^2}$; and the equation becomes $-\frac{w^2dv}{\sqrt{w^2dv^2 + (1-v^2)dw^2}} = \text{F}w$, or

$w^4dv^2 = \text{F}w^2(w^2dv^2 + (1-v^2)dw^2)$, or $\frac{dv}{\sqrt{1-v^2}} = \frac{\text{F}w.dw}{w\sqrt{w^2 - \text{F}w^2}}$,

an equation in which the variables are separated. (Paoli, Elementi d'Algebra, tom. ii. p. 152.)

4. *Clairaut's Form.*

$y = px + fp$, where $p = \frac{dy}{dx}$ and fp contains only p.

Differentiating the proposed equation, dy, which $= pdx$,

$= pdx + xdp + \frac{d.fp}{dp}dp$; wherefore

$0 = dp\left\{x + \frac{d.fp}{dp}\right\}$; or $dp = 0$, and $x + \frac{d.fp}{dp} = 0$; of

which the second, eliminating p by means of the proposed equation, will give a solution; but this is not the required primitive, since it does not contain an arbitrary constant. From the first we have $p = c$, and consequently the required primitive is $y = cx + fc$.

Hence all equations included under Clairaut's form are integrated by substituting a constant for the fluxional coefficient p.

Ex. 1. $ydx - xdy = b\sqrt{dx^2 + dy^2}$.

Here $y = xp + b\sqrt{1 + p^2}$, and the required fluent is $y = cx + b\sqrt{1 + c^2}$. If this equation be differentiated, and c be eliminated, there will result the proposed equation.

Also, to obtain another solution, we have

$x + \frac{bp}{\sqrt{1+p^2}} = 0$, or $p = -\frac{x}{\sqrt{b^2 - x^2}}$ and

$\sqrt{1+p^2} = \frac{b}{\sqrt{b^2-x^2}}$; and by substitution, $y = \sqrt{b^2-x^2}$,

or $x^2 + y^2 = b^2$. If this be differentiated, we have

$p = -\frac{x}{y}$; and substituting this value of p in the example, it is satisfied.

These two solutions are essentially different; the first is the equation of a right line, and the second belongs to a circle.

This example contains the solution of the following problem. 'Required the curve in which the perpendicular drawn from the origin of the co-ordinates upon the tangent is a constant quantity.'

Ex. 2. $ydx - xdy = \frac{a(dx^2+dy^2)}{dx}$.

Here $y = xp + a(1+p^2)$; and the complete primitive is $y = cx + a(1+c^2)$. Another solution is $x^2 = 4a(a-y)$.

Ex. 3. $ydx - nxdy = a\sqrt{dx^2+dy^2}$.

Here $y = nxp + a\sqrt{1+p^2}$ $\therefore dy$ or . . . . . .

$$pdx = npdx + nxdp + \frac{apdp}{\sqrt{1+p^2}} \therefore dx + \frac{nx}{n-1}\frac{dp}{p} \quad . \quad . \quad .$$

$$= -\frac{a}{n-1}\frac{dp}{\sqrt{1+p^2}} \therefore x = \frac{1}{p^{\frac{n}{n-1}}}\left\{c - \frac{a}{n-1}\int \frac{p^{\frac{n}{n-1}}dp}{\sqrt{1+p^2}}\right\}.$$

Suppose $n = \frac{2m+1}{2m}$; then $\frac{n}{n-1} = 2m+1$, and the form to be integrated becomes $\frac{p^{2m+1}dp}{\sqrt{1+p^2}} = du$; $\therefore$ (ch. i. 22, ex. 5.)

$$u = \sqrt{1+p^2}\left\{\frac{p^{2m}}{2m+1} - \frac{2mp^{2m-2}}{(2m+1)(2m-1)} + \ldots\right.$$

$$\left.\frac{2m(2m-2)p^{2m-4}}{(2m+1)(2m-1)(2m-3)} \ldots\right\}, \text{ and consequently}$$

$$x = \frac{c}{p^{2m+1}} - 2ma\sqrt{1+p^2}\left\{\frac{p^{-1}}{2m+1} - \frac{2mp^{-3}}{(2m+1)(2m-1)} + \ldots\right.$$

$$\frac{2m(2m-2)p^{-5}}{(2m+1)(2m-1)(2m-3)} - \cdots \Big\}.$$

If we suppose $2m = \infty$ in order that n may $= 1$, in which case the example coincides with ex. 1, we have

$$x = -a\sqrt{1+p^2}\left\{\frac{1}{p} - \frac{1}{p^3} + \frac{1}{p^5} - \cdots\right\} = -\frac{ap}{\sqrt{1+p^2}}$$

and $y = xp + a\sqrt{1+p^2}$, and as before a solution is $x^2 + y^2 = a^2$.

Ex. 4. $ydx - xdy = a(dx^3 + dy^3)^{\frac{1}{3}}$.

$y = xp + a(1+p^3)^{\frac{1}{3}}$ ∴ the complete primitive is · · ·

$y = cx + a(1+c^3)^{\frac{1}{3}}$.

Also to obtain another solution; we have

$$x + \frac{ap^2}{(1+p^3)^{\frac{2}{3}}} = 0, \text{ or } x = -\frac{ap^2}{(1+p^3)^{\frac{2}{3}}} \therefore \cdots$$

$$y = -\frac{ap^3}{(1+p^3)^{\frac{2}{3}}} + a(1+p^3)^{\frac{1}{3}} = \frac{a}{(1+p^3)^{\frac{2}{3}}} \therefore p^2 = -\frac{x}{y}. \cdots$$

But $y^3(1+p^3)^2 = a^3 \therefore 1 + p^3 = \dfrac{a^{\frac{3}{2}}}{y^{\frac{3}{2}}}$, or $p^3 = \dfrac{a^{\frac{3}{2}} - y^{\frac{3}{2}}}{y^{\frac{3}{2}}} \therefore \cdots$

$$\left(-\frac{x}{y}\right)^3 = \frac{(a^{\frac{3}{2}} - y^{\frac{3}{2}})^2}{y^3}, \text{ or } x^3 + (a^{\frac{3}{2}} - y^{\frac{3}{2}})^2 = 0.$$

5. *Required to integrate* $y = x\mathrm{P} + \mathrm{Q}$, *where* P *and* Q *are functions of* p.

Differentiating; dy, which $= pdx$, $= \mathrm{P}dx + xd\mathrm{P} + d\mathrm{Q}$; wherefore $(\mathrm{P} - p)dx + xd\mathrm{P} + d\mathrm{Q} = 0$, or

$dx + x \cdot \dfrac{d\mathrm{P}}{\mathrm{P}-p} = -\dfrac{d\mathrm{Q}}{\mathrm{P}-p}$; which is a linear equation, and consequently $x = e^{-\int\frac{d\mathrm{P}}{\mathrm{P}-p}}\left\{c - \int e^{\int\frac{d\mathrm{P}}{\mathrm{P}-p}}\dfrac{d\mathrm{Q}}{\mathrm{P}-p}\right\}$.

From this equation p must be eliminated by means of the proposed, and there will result the required primitive.

Ex. $bdx^2 - xdydx - ydy^2 = 0$, or $b - xp - yp^2 = 0$.

Here $y = -\frac{x}{p} + \frac{b}{p^2}$ $\therefore dy$, or

$$pdx = -\frac{dx}{p} + \frac{xdp}{p^2} - \frac{2bdp}{p^3} \text{ or } dx - \frac{xdp}{p(p^2+1)} = -\frac{2bdp}{p^2(p^2+1)};$$

which is a linear equation, where

$$\int \frac{dp}{p(p^2+1)} = l\frac{p}{\sqrt{p^2+1}}; \quad \ldots \ldots \ldots$$

$$\therefore x = \frac{p}{\sqrt{p^2+1}}\left\{\int \frac{-2bdp}{p^3\sqrt{p^2+1}} + c\right\} \quad \ldots \ldots$$

$$= \frac{p}{\sqrt{p^2+1}}\left\{\frac{b\sqrt{p^2+1}}{p^2} - bl\frac{1+\sqrt{p^2+1}}{p} + c\right\} \ldots$$

$$= \frac{b}{p} + \frac{cp}{\sqrt{p^2+1}} - \frac{bp}{\sqrt{p^2+1}} l\frac{1+\sqrt{p^2+1}}{p}; \text{ from which}$$

p is to be eliminated by means of $b - xp - yp^2 = 0$, and the required equation will result.

6. PRAXIS.

1. $dy^2 - bxdx^2 = 0$ $\therefore (y-a)^2 = \frac{4bx^3}{9}$.

2. $dy^2 = dxdy + x^2dx^2$ $\therefore y + a = \frac{x}{2} \pm \int \sqrt{x^2 + \frac{1}{4}}dx$.

3. $xdy^2 - ydxdy - \frac{y^2dx^2}{a} = 0$ $\therefore \ldots \ldots \ldots$

$$l\frac{y^2}{x} = \pm\int \frac{\sqrt{4x+a}}{a^{\frac{1}{2}}x}dx + c.$$

4. $y = \frac{xdy^2}{dx^2} + \frac{dy}{dx}$ $\therefore x = \frac{1}{(p-1)^2}\left\{c - p + lp\right\}$, from which p is to be eliminated by means of $y = xp^2 + p$.

5. $ydx - x\sqrt{dy^2 + dx^2} = 0$ $\therefore \ldots \ldots \ldots$

$p(p + \sqrt{1+p^2}) = l\frac{p+\sqrt{1+p^2}}{a^2}$, from which p may be eliminated by means of $\frac{y}{x} = \sqrt{1+p^2}$.

6. $xdx + ady = b\sqrt{dx^2 + dy^2}$ $\therefore \ldots \ldots \ldots$

$$\left.\begin{aligned} x &= b\sqrt{1+p^2} - ap \\ y &= bp\sqrt{1+p^2} - ap^2 - bf\sqrt{1+p^2}dp + \tfrac{1}{2}ap^2 \end{aligned}\right\}$$

7. $dy^3 - dx^3 = 0$. The primitive is of the form
$y^3 + x^3 - ay^2 - bxy - gx^2 + ey + fx + c = 0$.

7. SINGULAR SOLUTIONS.

It appears from Clairaut's form, art. 4, that a fluxional equation sometimes admits of a solution which is independent of its complete primitive or fluent. This is called a *singular solution;* and it must be distinguished from a *particular solution*, which is a particular case of the complete primitive.

8. *Def.* A *singular solution* is one which, when differentiated, satisfies the proposed equation, and is not included in its complete primitive.

Thus, of $y = xp + b\sqrt{1 + p^2}$ (art. 4, ex. 1), $x^2 + y^2 = b^2$ is a singular solution; for, differentiating, it satisfies the example, and yet is not included in the primitive, $y = ax + b\sqrt{1 + a^2}$.

9. Fluxional equations sometimes admit of other solutions than those which have been already noticed. If in the proposed equation $Pdx + Qdy = 0$, $F(x, y)$ is a factor both of P and of Q, it is manifest that the equation $F(x, y) = 0$ satisfies it. Equations of this kind are merely *algebraical solutions*, and do not form part of our subject. They are not singular solutions; because, *when differentiated*, they do not satisfy the proposed equation.

10. Singular solutions owe their origin to the nature of symbolical language. The symbols we use being more general than their archetypes, results are introduced into the calculation which were not contemplated in the original hypothesis. Thus, the first derivative is supposed to be obtained by eliminating a constant by means of the fluxion of the primitive; but if a be supposed to be variable, the same equation will in some cases result; and if, in order to render the results as general as possible, we differentiate the primitive on the supposition that a is a variable, we shall trace these solutions to their true origin.

11. *Required to find the singular solution, of a fluxional equation whose primitive is known.*

Let $v = F(x, y, a) = 0$ be the primitive clear of radicals; then if a be constant, we have

$\frac{dv}{dx} = \frac{dv}{dx} + \frac{dv}{dy}\frac{dy}{dx} = 0$; but if a be variable,

$\frac{dv}{dx} = \frac{dv}{dx} + \frac{dv}{dy}\frac{dy}{dx} + \frac{dv}{da}\frac{da}{dx} = 0.$

Now when a corresponds to a singular solution, these equations are simultaneous; because, when combined with the primitive, they each produce the same result, viz. the proposed fluxional equation; and consequently we have

$\frac{dv}{da}\frac{da}{dx} = 0.$

This is satisfied either by $\frac{da}{dx} = 0$, or by $\frac{dv}{da} = 0$; of which the first gives $a =$ a constant quantity, and therefore belongs to a particular solution. The second is $\frac{dv}{da} = 0$, an equation from which a may be obtained in terms of (x, y); and if this value of a be substituted in the primitive, there will result an equation which does not contain an arbitrary constant; and which may be a singular solution.

If the resulting value of a be constant; or even if it be such a function of both or of either of the variables, that the resulting equation is deducible from the primitive by assigning a value to a independent of x and y; then it is a particular, and not a singular solution.

There is one exception which we shall not consider in this article: if $\frac{dv}{dx}$ and $\frac{dv}{dy}$ be such that they may both become infinite by assigning a particular value to a, then the two equations may be simultaneous and yet $\frac{dv}{da}\frac{da}{dx}$ may not $= 0$.

Hence the rule. 'Let the primitive, clear of radicals, be $v = 0$: differentiate this on the hypothesis that a is variable. Let $B = 0$ be one of the factors of $\frac{dv}{da} = 0$: eliminate a by means of $B = 0$ and of $v = 0$; and the resulting equation, if not a particular, is a singular solution.'

Cor. Arranging the terms of the primitive according to the powers of a, $\frac{dv}{da} = 0$ is its limiting equation (Alg. 319);

and is therefore the condition upon which a has two *equal* roots.

In the article we suppose the primitive to have been cleared of radicals; for, if it be solved with respect to a, and be under the form $a = \varphi(x, y)$, or $a - \varphi(x, y) = 0 = w$, we shall have $\frac{dw}{da} = 1$, by means of which we cannot eliminate a from $w = 0$. By means of the two following articles we are enabled to find the singular solution without clearing the primitive of radicals.

12. *If the primitive be solved with respect to its arbitrary constant, the partial fluxional coefficients of the arbitrary constant corresponding to a singular solution are each infinite.*

For let the primitive be $\mathrm{F}(x, y, a) = 0 = v$; and when solved with respect to a, let it take the form $a = \varphi(x, y)$; then it is evident, that if this value of a be substituted in $\mathrm{F}(x, y, a) = 0$, the equation becomes *identical;* or one which obtains whatever values be assigned to x or to y. Hence it is a function of *two* independent variables; and we have

$$\left.\begin{aligned} 0 &= \frac{dv}{dx} + \frac{dv}{da}\frac{da}{dx} \\ 0 &= \frac{dv}{dy} + \frac{dv}{da}\frac{da}{dy} \end{aligned}\right\}.$$ But when a corresponds to a singular solution, $\frac{dv}{da} = 0$; wherefore $\frac{da}{dx} = \infty$ and $\frac{da}{dy} = \infty$.

It is here supposed that $\frac{dv}{dx}$ and $\frac{dv}{dy}$ are not $= 0$.

This proposition is not true, è converso, for the resulting solution may give $\varphi(x, y)$, when either x or y is eliminated, $=$ a constant quantity; in which case it is a particular, and not a singular solution.

Cor. 1. If the equation $a = \varphi(x, y)$ contain a radical, a singular solution *may be* obtained by making the radical $= 0$.

For radicals, when differentiated, come into the denominator, and consequently the resulting values of $\frac{da}{dx}$ and $\frac{da}{dy}$ ... will be infinite.

Cor. 2. Hence the rule. 'Let the primitive be solved with respect to a; then if a contain a radical, make it $= 0$, and the equation, if not a particular, is a singular solution.'

Cor. 3. So long as the radical remains, the roots of a are necessarily unequal; but when it is made to disappear, if either x or y be eliminated by means of $a = \varphi(x, y)$ and the original primitive $\mathrm{F}(x, y, a) = 0 = v$, the resulting form of $v = 0$ will exhibit a with two equal roots; which agrees with art. 11, cor.

13. When a rises only to the first degree in the primitive, it is of the form $\mathrm{A}a + \mathrm{P} = 0 = w$, where A and P are functions of x, y which do not contain a.

First suppose A not to be a factor of P; then since $a = -\frac{\mathrm{P}}{\mathrm{A}}$, from $\frac{dw}{da} = \mathrm{A} \doteq 0$, there results $a = \infty$; which gives a particular solution, viz. that case of the primitive in which the arbitrary constant is supposed to be infinite.

Next, let A be a factor of P; then, since the proposed fluxional equation is equivalent to $0 = \mathrm{P}d\mathrm{A} - \mathrm{A}d\mathrm{P}$ (ch. 3, 6), we have $\mathrm{A} = 0$ a solution; and to determine whether it is a singular solution, eliminate either x or y from $\mathrm{A} = 0$ and the primitive; and it is a singular solution or not, according as the resulting value of a is variable or constant.

If a should appear under the form $\frac{0}{0}$, A and P have a common factor which is independent of a (vol. i. ch. 5, 5). Let it be $\mathrm{B} = 0$, and suppose $\mathrm{P} = m.\mathrm{B}$ and $\mathrm{A} = n.\mathrm{B}$; then the primitive becomes $(m + na)\mathrm{B} = 0$; which shows that $\mathrm{B} = 0$ is only a factor of the primitive, and is neither a particular nor a singular solution.

14. *The value of the factor which renders an equation an exact fluxion corresponding to a singular solution is always infinite.*

Since an exact fluxion is necessarily of the first degree, let the equation be solved with respect to p and be under the form $\mathrm{P}dx + \mathrm{Q}dy = 0$; and let z be the factor such that $z(\mathrm{P}dx + \mathrm{Q}dy) = 0 = dw$, an exact fluxion; then, integrating, $w = a$ is the primitive, where a is the arbitrary constant.

Differentiating on the supposition that a corresponds to a singular solution, we have $da = dw = z(\mathrm{P}dx + \mathrm{Q}dy)$,

But $Pdx + Qdy = 0$ and da is not $= 0$ (ex hyp.); wherefore $z = \infty$.

The converse of this proposition does not hold good; because z may be infinite, and yet da may $= 0$.

Otherwise. It has been shown, ch. 4, art. 22, that if the primitive, when solved with respect to its arbitrary constant, be $a = F(x, y) = w$, the factor which renders the proposed equation, when under the form $p + f(x, y) = 0$, an exact fluxion is $\frac{dw}{dy}$; which $= \frac{da}{dy}$, if a be considered as variable; which is infinite when a belongs to a singular solution.

This and all the preceding propositions are equally applicable to equations of a higher order than the first; for they are derived from their first primitives by the same process of differentiation combined with elimination.

Before we proceed to the investigation of those properties of singular solutions, which can be ascertained without the aid of the complete primitive, we shall illustrate the preceding articles by a few examples.

15. EXAMPLES.

Ex. 1. $ydy^2 + 2xdydx = ydx^2$, or $yp^2 + 2xp = y$.

The complete primitive is $a^2 + 2ax - y^2 = 0 = v$ (art. 1, ex. 2); $\therefore \frac{dv}{da} = 2a + 2x = 0$, or $a = -x$. If this be substituted in the primitive, it becomes $x^2 + y^2 = 0$, a *singular* solution; because when differentiated, it gives $p = -\frac{x}{y}$, which substituted in $yp^2 + 2xp = y$, renders it identical.

Solving the primitive, we have $a = -x \pm \sqrt{x^2 + y^2}$. $\therefore \frac{da}{dx}$ and $\frac{dx}{dy}$ are both infinite, when $x^2 + y^2 = 0$. Also eliminating y^2 from the primitive by means of $x^2 + y^2 = 0$, it becomes $a^2 + 2ax + x^2 = 0$, where a has two equal roots. Lastly, the proposed equation is the same as $p = -\frac{x + \sqrt{x^2+y^2}}{y}$, or $ydy + xdx + \sqrt{x^2 + y^2}dx = 0$, and the factor which renders it an exact fluxion is

$\frac{1}{\sqrt{x^2+y^2}}$; which is infinite when $x^2 + y^2 = 0$.

Ex. 2. Let the primitive be
$(x^2 + y^2 - b)(y^2 - 2ay) + (x^2 - b)a^2 = 0 = v$.

If this be solved with respect to a, there results

$$a = \frac{y(x^2+y^2-b) \pm y^2\sqrt{x^2+y^2-b}}{x^2-b}$$ where $\sqrt{x^2+y^2-b}$ is

the only radical; but $x^2 + y^2 - b = 0$ is not a singular solution, for it results from supposing in the original primitive that $a = 0$.

Ex. 3. Let the primitive be $y^2-2ay+x^2-a^2=0=v$.

Here the fluxional equation is $(x^2-2y^2)p^2 - 4xyp = x^2$.

Also $\frac{dv}{da} = 2a + 2y = 0$, or $a = -y$; which substituted in the primitive, gives $2y^2 + x^2 = 0$, a singular solution.

Solving the primitive, it becomes $a = -y \pm \sqrt{2y^2 + x^2}$

$\therefore \frac{da}{dx}$ and $\frac{da}{dy}$ are infinite when $2y^2 + x^2 = 0$. Also, eliminating x^2 from the primitive by means of $2y^2 + x^2 = 0$, there results $a^2+2ay+y^2 = 0$ where a has two equal roots.

Ex. 4. Let the primitive be $x^2 - 2ay - a^2 - b = 0 = v$.

The fluxional equation is $\sqrt{x^2 + y^2 - b}dy - ydy = xdx$.

Also $\frac{dv}{da} = -2y - 2a = 0$, or $a = -y$; which substituted in the primitive, gives $x^2 + y^2 - b = 0$, which is a singular solution; because when differentiated, it satisfies . . . $\sqrt{x^2 + y^2 - b}dy = xdx + ydy$.

16. We have already seen instances of the application of singular solutions to the Theory of Curves in the first volume, ch. 11, art. 17.

In those examples, if $F(x, y, a) = 0$ is the complete primitive, it may be considered as representing an infinite number of curves of the same species formed by a considered as a *fluent*. Their intersections, which may be considered as the loci of particular primitives in each of which a has a certain value, form the tangential curve. In this latter curve, which is the singular solution, a has not a certain fixed value, but is a function of the co-ordinates at the point of contact. This difference constitutes the prin-

ciple upon which the examples of ch. 11, arts. 19 and 20, are solved.

17. *The fluxion of an equation of the first order, which has a singular solution, may always be put under the form of two factors; one of which is an equation of the second order which belongs to the complete primitive; and the other, an equation of the first order which belongs to the singular solution.*

It appears from the theory of arbitrary constants (ch. 3, 2), and from this chapter, art. 11, that when $\text{F}(x, y, a) = 0 = v$ is the complete primitive, the proposed fluxional equation has been obtained by eliminating a constant from $v = 0$ and $\frac{\dot{dv}}{dx} = 0$; but when it is a singular solution, by eliminating a variable from $v = 0$ and $\frac{dv}{da} = 0$.

Let the value of a as deduced from $\frac{\dot{dv}}{dx} = 0$ be under the form $a = \varphi(x, y, p)$, which for the sake of convenience denote by φ; then the proposed fluxional equation is $\text{F}(x, y, \varphi) = 0 = u$.

Now, differentiating both $v = 0$ and $u = 0$, we have

$$\left.\begin{aligned} \frac{\dot{dv}}{dx} = 0 &= \frac{dv}{dx} + \frac{dv}{dy}\frac{dy}{dx} \\ \frac{\dot{du}}{dx} = 0 &= \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} + \frac{du}{d\varphi}\frac{\dot{d\varphi}}{dx} \end{aligned}\right\}. \quad \text{But } \frac{du}{dx} \text{ and } \frac{du}{dy} \text{ are}$$

manifestly equivalent to $\frac{dv}{dx}$ and $\frac{dv}{dy}$: they are under a different form; for where a enters in the former quantities, φ will enter in the latter. Hence $\frac{\dot{du}}{dx} = 0$ is satisfied by

$$\frac{du}{d\varphi}\frac{\dot{d\varphi}}{dx} = 0, \textit{ i. e.} \text{ by } \frac{du}{d\varphi} = 0 \text{ and by } \frac{\dot{d\varphi}}{dx} = 0.$$

First, let $\frac{\dot{d\varphi}}{dx} = 0$. This is an equation of the second order, and one of its first primitives is $\varphi = a$, a constant quantity; or its two first primitives are $\text{F}(x, y, \varphi) = 0 = u$ and $\varphi = a$; from which if φ be eliminated, which is tanta-

mount to the eliminating of p, there results $F(x, y, a)$, the original primitive; or $\frac{\dot{d}\varphi}{dx} = 0$ is an equation of the second order, which belongs to the complete primitive.

Next, let $\frac{du}{d\varphi} = 0$. This is of the first order: it satisfies $\frac{\dot{d}u}{dx} = 0$ independently of $\frac{\dot{d}\varphi}{dx} = 0$; and does not contain an arbitrary constant.

If then p be eliminated from it by means of $u = 0$, which is effected as before by eliminating φ from the same two equations, there will result an independent primitive which does not contain a; and which is therefore the singular solution.

Cor. The factor which gives the singular solution, and which is an equation of the first order, will not satisfy those equations of the complete primitive which are of a higher order than the first; for $\frac{du}{d\varphi} = 0$ is independent of $\frac{\dot{d}\varphi}{dx} = 0$.

Ex. Let the complete primitive be $a^2 + 2ax - y^2 - b = 0 = v$; whose derivative is $y^2p^2 + 2xyp - y^2 - b = 0$.

Differentiating the primitive, we have $\frac{\dot{d}v}{dx} = 2a - 2yp = 0$, or $a = yp = \varphi$; $\therefore \frac{\dot{d}\varphi}{dx} = 0 = p^2 + yq$. Also the primitive is equivalent to $(\varphi^2 + 2x\varphi - y^2 - b) = 0 = u$ $\therefore$ . . . $\frac{du}{d\varphi} = 2\varphi + 2x = 0$, or $yp + x = 0$.

The fluxion of the derivative is $yp^3 + y^2pq + xp^2 + xyq$, which may be put under the form of $(p^2+yq)(yp + x) = 0$; where $p^2 + yq = 0$ is the second derivative of the primitive; and $yp + x = 0$, or $p = -\frac{x}{y}$, when substituted in the proposed, gives $x^2+y^2-b = 0$, the singular solution.

18. *If a fluxional equation be solved with respect to*

$\frac{dy}{dx}$ *or* p, *both the fluxional coefficients of* p *corresponding to a singular solution shall be infinite.*

For let the equation be $p + f(x, y) = 0$. Also let the primitive be $\text{F}(x, y, \varphi) = 0 = u$; where φ denotes $\varphi(x, y, p) = a$. Now if the value of p, viz. $-f(x, y)$ be substituted in φ, u will become an identical equation; and consequently we have (vol. i. ch. 4. 31),

$$0 = \frac{du}{dx} + \frac{du}{d\varphi}\frac{d\varphi}{dx}$$
$$= \frac{du}{dx} + \frac{du}{d\varphi}\left\{\frac{d\varphi}{dx} - \frac{d\varphi}{dp}\frac{dp}{dx}\right\}$$
$$= \frac{du}{dx} + \frac{du}{d\varphi}\frac{d\varphi}{dx} - \frac{du}{d\varphi}\frac{d\varphi}{dp}\frac{dp}{dx}$$
$$\text{and } 0 = \frac{du}{dy} + \frac{du}{d\varphi}\frac{d\varphi}{dy} - \frac{du}{d\varphi}\frac{d\varphi}{dp}\frac{dp}{dy}.$$

But in the case of a singular solution $\frac{du}{d\varphi} = 0$. Also $\frac{d\varphi}{dx}$, $\frac{d\varphi}{dy}$ and $\frac{d\varphi}{dp}$ are not necessarily infinite; wherefore, we have $\frac{dp}{dx} = \infty$ and $\frac{dp}{dy} = \infty$.

Cor. Equations, whose primitives are homogeneous, or which may be rendered homogeneous by proper substitutions, do not admit of a singular solution (ch. 3, art. 9).

19. *Required to determine whether a given solution is a particular or a singular solution of an equation whose primitive is unknown.*

The characteristick property of a singular solution is that there is not any independent value of the arbitrary constant which can render the complete primitive identical with it.

Let $y = \text{x}$ be the given solution; and $y = \text{v}$, the complete primitive; where the two equations are supposed to be solved with respect to y; and x is a known, and v an unknown function of x.

Let c be the arbitrary constant; then, if $y = \text{x}$ is a *particular* solution, there is some quantity c independent of

x, which substituted for c will reduce $y = \mathrm{v}$ to $y = \mathrm{x}$. And since the supposition of $\mathrm{c} = c$ gives $\mathrm{v} = \mathrm{x}$, we have $\mathrm{v} - \mathrm{x} = 0$ when $\mathrm{c} - c = 0$; and consequently we may assume $\mathrm{v} - \mathrm{x} = \mathrm{v}'(\mathrm{c} - c)^{\alpha} + \mathrm{v}''(\mathrm{c} - c)^{\beta} + \cdots$, where v', v'', $\cdots$ are functions of x independent of $\mathrm{c} - c$; and the indices α, β, $\cdots$ are positive. It is also supposed that the terms are arranged in an ascending order. Substituting $h = (\mathrm{c} - c)^{\alpha}$, we have $\mathrm{v} - \mathrm{x} = \mathrm{v}'h + \mathrm{v}''h^{\mu} + \cdots$ where $\mu = \dfrac{\beta}{\alpha}$; or $\mathrm{v} = \mathrm{x} + \mathrm{v}'h + \mathrm{v}''h^{\mu} + \cdots = \mathrm{x} + k$, if $k = \mathrm{v}'h + \mathrm{v}''h^{\mu} + \cdots$; which may therefore be considered as the developement of the *complete* value of y, when $y = \mathrm{x}$ is a *particular* solution.

Let the proposed fluxional equation, when solved with respect to $\dfrac{dy}{dx}$, be $\dfrac{dy}{dx} = p$ or $dy = pdx$. This equation is (ex hyp.) satisfied by $y = \mathrm{x}$; but when $y = \mathrm{x}$ is a *particular* solution, it is satisfied by the complete primitive, viz. $y = \mathrm{x} + k$, where k is independent of x. To find then what it becomes when $y = \mathrm{x} + k$, let the corresponding value of p be $\mathrm{p} + \mathrm{p}'k^{m} + \mathrm{p}''k^{n} + \cdots$ The terms are here arranged according to the ascending powers of k; and k does not enter into the first term; for when $k = 0$, or $y = \mathrm{x}$, $p = \dfrac{dy}{dx}$, or it does not vanish, which shows that $\mathrm{p} = p$; nor are any of the indices negative, for when $k = 0$, p is not infinite. By this assumption then we have $d(\mathrm{x} + k) = (\mathrm{p} + \mathrm{p}'k^{m} + \mathrm{p}''k^{n} + \cdots)dx$; but $d\mathrm{x} = \mathrm{p}dx$, wherefore $dk = (\mathrm{p}'k^{m} + \mathrm{p}''k^{n} + \cdots)dx$; and replacing k by its value, there results

$$h d\mathrm{v}' + h^{\mu} d\mathrm{v}'' + \cdots = \left\{ \begin{array}{l} \mathrm{p}'h^{m}(\mathrm{v}' + \mathrm{v}''h^{\mu-1} + \cdots)^{m} \\ + \mathrm{p}''h^{n}(\mathrm{v}' + \mathrm{v}''h^{\mu-1} + \cdots)^{n} \\ + \cdots \end{array} \right\} dx \ (\alpha);$$

an equation from which, when $y = \mathrm{x}$ is a particular solution, v', v'', $\cdots$ may be determined independently of h. When this is not possible, $y = \mathrm{x}$ is a singular solution.

Now first equating the two terms which have the least indices, we have $h d\mathrm{v}' = \mathrm{p}'\mathrm{v}'^{m} h^{m} dx$; which, if $m = 1$, becomes $d\mathrm{v}' = \mathrm{p}'\mathrm{v}' dx$, or $\mathrm{v}' = e^{\int \mathrm{p}' dx}$. Similarly, on the same supposition, all the remaining terms v'', v''', $\cdots$ may be determined independently of h.

Next let $m > 1$. Here we cannot equate the two first terms; but if we suppose that $dv' = 0$, or $v' =$ constant, or $v' = 1$, for v' is not to contain an arbitrary constant; and also that $\mu = m$, we shall have $dv'' = p'dx$, or $v'' = \int p'dx$; and all the remaining terms may be determined in the same manner.

But if $m < 1$, we cannot by any supposition whatever make the two series identical; for μ and all the succeeding indices are (ex hyp.) greater than unity. In this case, therefore, the equation $y = x$ does not admit of any such quantity as k, and it is a singular and not a particular solution.

Hence the rule. 'In the equation $\frac{dy}{dx} = p$, substitute $x + k$ for y, and develope the resulting value of $\frac{dy}{dx}$ in a series ascending by the powers of k. Let the series be $P + P'k^m + P''k^n + \cdots$; then, if $m =$ or > 1, $y = x$ is a particular solution; but if $m < 1$, it is a singular solution.'

Cor. 1. The value of $\frac{dp}{dy}$ corresponding to a singular solution is infinite (vol. i. ch. 5, 4).

Cor. 2 Singular solutions belong to those cases in which p fails to be developable by means of Taylor's theorem.

Cor. 3. The value of $\frac{dp}{dx}$ corresponding to a singular solution is also infinite.

For let the proposed equation clear of radicals be $F(x, y, p) = 0 = u$; then, if it correspond to a singular solution, u is a function of two independent variables x and y; and consequently we have

$$\left.\begin{aligned} 0 &= \frac{du}{dx} + \frac{du}{dp}\frac{dp}{dx} \\ 0 &= \frac{du}{dy} + \frac{du}{dp}\frac{dp}{dy} \end{aligned}\right\}. \quad \text{But } \frac{dp}{dy} = \infty \text{ ; wherefore } \frac{du}{dp} = 0, \text{ and}$$

consequently $\frac{dp}{dx} = \infty$.

20. In the preceding investigation it is supposed that y is a function of x; but if y does not enter into the given solution, *i. e.* if the given solution is under the form $x = a$, we must suppose x to be a function of y. Let the proposed

fluxional equation be $dx = p_1 dy$, where $p_1 = \frac{dx}{dy}$; then it may be shown, as in the preceding article, that if $a + k$ be substituted for x, and the resulting value of $\frac{dx}{dy}$ be developed in a series ascending by the powers of k of the form $P_1 + P'_1 k^m + P''_1 k^n + \cdots$, $x = a$ is a singular solution only when $m < 1$.

21. It appears then that whether we proceed by Lagrange's method (arts. 17 and 18), or by Euler and Laplace's (art. 19), we arrive at the same property of singular solutions, viz. that they render infinite both the fluxional coefficients of $p = f(x, y)$. They will therefore be found among the radicals contained in this value of p.

To apply this property, it is necessary to solve the proposed equation with respect to p; but by means of the following article this may be avoided.

22. *Required to find the singular solution of a fluxional equation whose primitive is unknown.*

Let $F(x, y, p) = 0 = u$ be the equation clear of radicals and fractions; then, when it corresponds to a singular solution, u is a function of two independent variables (x, y), and it may be shown as in art. 19, cor. 3, that $\frac{du}{dp} = 0$.

Eliminate then p from $\frac{du}{dp} = 0$ and $u = 0$, and there will result the required singular solution.

It is supposed in this as in art. 19, cor. 3, that the resulting solution renders $\frac{du}{dx}$ and $\frac{du}{dy}$ finite; otherwise $\frac{du}{dp}$ may not $= 0$.

23. *The second fluxional coefficient corresponding to a singular solution appears under the form of* $\frac{0}{0}$.

For differentiating $u = 0$ totally, we have

$0 = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dp}dp$, or

$dp = -\dfrac{\frac{du}{dx}dx + \frac{du}{dy}dy}{\frac{du}{dp}} = \dfrac{0}{0}$, when $\dfrac{du}{dp} = 0$; for then

$\dfrac{du}{dx}dx + \dfrac{du}{dy}dy = 0$; wherefore q, which $= \dfrac{dp}{dx}$, is under the form of $\dfrac{0}{0}$.

Hence the following rule. 'Differentiate the proposed equation $u = 0$; and find the value of q. Let $q = \dfrac{\mathrm{P}}{\mathrm{Q}}$; then $u = 0$, $\mathrm{P} = 0$, and $\mathrm{Q} = 0$, are three equations which contain p; and if there be any the same solution which satisfies all three, when p is eliminated, it is a singular solution.'

The truth of this proposition also appears from ch. 3, art. 13; for if the second and the higher fluxional coefficients do not become indeterminate, the value of y in terms of x will involve arbitrary constants, and consequently will be its complete value. (Lagrange, Calc. des Fonctions, p. 223.)

24. All that is known of the properties of singular solutions and of the theory of arbitrary constants upon which they are founded, is due to the German and French mathematicians. If the reader is desirous of further information, I must refer him to their works. The principal authors are Euler, Lagrange, Legendre, and Poisson. In his Calcul des Fonctions, Lagrange has dedicated the fourteenth and the three following chapters to this subject; and Poisson has made some important additions in the 'Journal de l'E'cole Polytechnique, 13e cahier, p. 113.' This branch of the subject is as important as any other; for the problem which has given rise to the fluxional equation frequently depends upon the singular solution, and not upon the complete primitive.

25. EXAMPLES.

Ex. 1. $y = xp + b\sqrt{1 + p^2}$ (art. 4, ex. 1).

Here $-y + xp + b\sqrt{1 + p^2} = 0 = u$ . . . . . .

$\therefore \frac{du}{dp} = x + \frac{bp}{\sqrt{1+p^2}} = 0$, or $p = -\frac{x}{\sqrt{b^2-x^2}}$, and $\sqrt{1+p^2} = \frac{b}{\sqrt{b^2-x^2}}$; and eliminating p by means of $\frac{du}{dp} = 0$, $x^2 + y^2 = b^2$ is a singular solution not contained in the complete primitive $y = cx + b\sqrt{1+c^2}$.

Ex. 2. $y = xp + fp$. (Clairaut's form.)

The fluxion is under the form of

$\left\{x + \frac{d.fp}{dp}\right\} q = 0$ (art. 4); of which $q = 0$ gives $p = c$, or $y = cx + fc$, the complete primitive; and

$x + \frac{d.fp}{dp} = 0$, which is the same as $\frac{du}{dp} = 0$, will give the singular solution, if p be eliminated by means of $y = xp + fp$.

Hence all equations included under Clairaut's form have a singular solution.

Ex. 3. $(a + y)\frac{dx}{dy} = (x + y) - x\frac{dy}{dx}$, or $a + y = (x + y)p - xp^2$. (Cambridge Probs. p. 405.)

Since $dy = pdx$, the fluxion is under the form of $0 = (x + y)q - 2xpq = (x + y - 2xp)q$; of which $q = 0$ gives $p = c$, or $a + y = c(x + y) - c^2x$ the complete primitive; and to find the singular solution, we have $x + y - 2xp = 0$, or $p = \frac{x+y}{2x}$; and consequently the singular solution is $a + y = \frac{(x+y)^2}{2x} - \frac{(x+y)^2}{4x} = \frac{(x+y)^2}{4x}$, or $x - y = 2\sqrt{ax}$.

Ex. 4. $p^n + Ap^{n-1} + Bp^{n-2} \cdots + Gp + H = 0 = u$, where A, B, $\cdots$ G, H are constant quantities.

Here $\frac{du}{dp} = np^{n-1} + (n-1)Ap^{n-2} \cdots + G$; and the values of p in $\frac{du}{dp} = 0$ are the limits between the roots of $u = 0$, and we cannot eliminate p. If $u = 0$ has *equal* roots, we may eliminate p; but since p is constant, the resulting equation is a particular solution.

It is observable that in this case $\frac{dp}{dy}$ appears under the form of $\frac{0}{0}$; since $\frac{du}{dp} = 0$ and $\frac{du}{dy} = 0$. (Art. 19, cor. 3)

Ex. 5. $a^2dy + y^2dx = a^2dx + xydx$.

Here $y = x$ is a solution. And to find whether it is a singular solution, we have $\frac{dy}{dx} = \frac{a^2 + xy - y^2}{a^2} = p$. Substitute $x+k$ for y; then p becomes $1 + \frac{x(x+k) - (x+k)^2}{a^2}$ $= 1 - \frac{xk + k^2}{a^2}$; or $y = x$ is not a singular solution.

To find the complete primitive; we have

$a^2(dy - dx) = (xy - y^2)dx$, or $\frac{a^2d(y-x)}{y-x} = -ydx$. Substitute $v = \frac{1}{y-x}$; then $a^2dv - vxdx = dx$; and integrating, $v = e^{\frac{x^2}{2a^2}} \left\{ c + \frac{1}{a^2} \int e^{-\frac{x^2}{2a^2}} dx \right\}$ which, when $c = \infty$, gives $v = \infty$, or $y = x$ a *particular* solution.

Ex. 6. $\frac{dy}{dx} = b(y - a)^n$.

Here $y = a$ is a solution. And to find whether it is a particular or a singular solution, let y become $a + k$; then $\frac{dy}{dx}$ becomes bk^n; or $y = a$ is a singular solution only when $n < 1$.

The complete primitive is $\frac{(y-a)^{1-n}}{1-n} = bx + c$. If $n < 1$, $\frac{dp}{dy}$, which $= \frac{nb}{(y-a)^{1-n}} = \infty$ when $y = a$; but if $n > 1$; or if $n = 1$, in which case $\frac{dp}{dy} = \frac{b}{0^0} = b$ (vol. i. ch. 5, ex. 10, cor. 2), in either case $\frac{dp}{dy}$ is finite, and $y = a$ is not a singular solution.

Generally, if $\frac{dy}{dx} = \sqrt{\text{Y}}$ where Y is a function of y, we have $\frac{dp}{dy} = \frac{1}{2\sqrt{\text{Y}}}\frac{d\text{Y}}{dy}$. Let $(y-a)^n$ be a factor of Y; then $\left(\frac{dp}{dy}\right) = \infty$, and there is a singular solution only when $\frac{n}{2} < 1$.

Ex. 7. $dx(1-y^m)^n = dy(1-x^m)^n$.

Here $y = x$ is a solution and $\frac{dy}{dx} = \frac{(1-y^m)^n}{(1-x^m)^n}$. Substitute $y = x + k$; then $\frac{dy}{dx}$ becomes $\left\{\frac{1-(x+k)^m}{1-x^m}\right\}^n \cdots$

$= \left\{\frac{1-x^m-mx^{m-1}k-\cdots}{1-x^m}\right\}^n = \left\{1-\frac{mx^{m-1}k}{1-x^m}\cdots\right\}^n \cdots$

$= 1 - \frac{nmx^{m-1}k}{1-x^m} + \cdots$; $\therefore$ $y = x$ is a *particular* solution.

Ex. 8. $e^x dx - \frac{ydy}{e^y} = dy - ydx$ (ch. 4, art. 11, ex. 9).

Here $y = e^x$ is a solution. Also $\frac{dy}{dx} = e^x = y$ $\therefore$ when y becomes $y + k$, p becomes $p + k$; or $y = e^x$ is a particular solution.

Ex. 9. $dy = \frac{xdx}{\sqrt{1-x^4}}$.

Here $\frac{dy}{dx}$ does not contain y. Substitute $p_1 = \frac{dx}{dy}$, then $p_1 = \frac{\sqrt{1-x^4}}{x} = \sqrt{x^{-2}-x^2}$. Substitute $\pm 1 + k$ for x^2; then p_1 becomes $\sqrt{(\pm 1+k)^{-1} - (\pm 1 + k)} \cdots$ $= \sqrt{-2k \pm k^2} - \cdots$ or $1 + x^2 = 0$ and $1 - x^2 = 0$ are both singular solutions, and they are not included in the complete primitive $x^2 = \sin.(2y + c)$.

Let $x^2 - \sin.(2y + c) = 0 = v$; then $\frac{dv}{dc} = -\cos.(2y + c) = 0$; from which we obtain

$2y + c = \frac{\pi}{2}$, or $\frac{3\pi}{2}$, or $\cdots$, and consequently $c = \frac{\pi}{2} - 2y$, or $= \frac{3\pi}{2} - 2y$, variable quantities; from which also it follows that these are singular solutions; and if they be substituted in the primitive, there will result as before $x^2 = \pm 1$.

Ex. 10. $(xdy - ydx)(xdy - 2ydx) + x^3dx = 0$.

Here $(xp - y)(xp - 2y) + x^3 = 0 = u \therefore \cdots$ $\frac{du}{dp} = x(xp - 2y) + x(xp - y) = 0 = 2xp - 3y$, or $xp = \frac{3y}{2} \therefore 0 = -\frac{y^2}{4} + x^3$, or $y = 2x^{\frac{3}{2}}$ is a singular solution.

Or thus: $\frac{du}{dx} = 0 = xq(xp - 2y) + (xq - p)(xp - y) + 3x^2$, or $2x^2pq - 3xyq - xp^2 + yp + 3x^2 = 0$, or $q = \frac{xp^2 - yp - 3x^2}{x(2xp - 3y)}$.

Assume $q = \frac{0}{0}$; or $xp^2 - yp - 3x^2 = 0$ and $2xp - 3y = 0$. Of these, the second gives $p = \frac{3y}{2x}$, which substituted in $u = 0$, gives $0 = -\frac{y^2}{4} + x^3$. If this value of p be also substituted in the first, it gives $\frac{9y^2}{4x} - \frac{3y^2}{2x} - 3x^2 = 0$, or $-\frac{y^2}{4} + x^3 = 0$, the same result as before; and consequently $-\frac{y^2}{4} + x^3 = 0$, or $y = 2x^{\frac{3}{2}}$ is a singular solution. The complete primitive is $y - ax = \frac{x^2}{a}$.

Ex. 11. $ydx - xdy = x\sqrt{dx^2 + dy^2}$.

Here $(y - xp)^2 - x^2(1 + p^2) = 0 = u \therefore \cdots$ $\frac{du}{dx} = 0 = -2(y - xp)xq - 2x(1 + p^2) - 2x^2pq \cdots$

$= -2xyq - 2x(1+p^2)$, or $q = -\frac{x(1+p^2)}{xy}$ which becomes $\frac{0}{0}$ when $x = 0$ unless $p = \infty$; but this is not a singular solution, since it is a particular case of the complete primitive $y^2 + x^2 = 2ax$ (art. 3, ex. 1), viz. that in which $a = \infty$.

The rule of art. 22 fails here, because $x = 0$ renders infinite $\frac{du}{dx}$, which $= -2yp - 2x$.

Ex. 13. $y^6dx^6 - xydxdy^5 - 2xydx^5dy + x^6dy^6 + x^2dx^4dy^2 = 0$.

Here $x^2p^6 - xyp^5 + x^2p^2 - 2xyp + y^2 = 0 = u$, and $y = x$ is a solution. To find whether it is a singular solution, we have $\frac{du}{dx} = 0 = xp^6 - yp^5 + (6x^2p - 5xyp^4 + 2x^2p - 2xy)q$,

or $q = \frac{yp^5 - xp^6}{6x^2p^5 - 5xyp^4 + 2x^2p - 2xy}$; and consequently $y = x$ is not a singular solution.

Ex. 14. $y - x\frac{dy}{dx} + \left\{\frac{x^2}{2} - \frac{dy}{dx}\right\}\frac{d^2y}{dx^2} + x\frac{d^2y^2}{dx^4} = 0$, or

$xq^2 + \frac{x^2 - 2p}{2}q - xp + y = 0 = u$.

This example and all equations of the second order may have two different kinds of singular solutions; the one, an equation of the first order whose primitive contains one arbitrary constant; and the other, a primitive which does not contain any arbitrary constant. The first is to be obtained by eliminating q by means of $u = 0$ and $\frac{du}{dq} = 0$; and the other, by eliminating both q and p by means of $u = 0$, $\frac{du}{dq} = 0$, and $\frac{du}{dp} = 0$; and the result, if it satisfy $u = 0$, is in either case a singular solution.

Here $\frac{du}{dq} = 2xq + \frac{x^2 - 2p}{2} = 0$, or $q = \frac{2p - x^2}{4x}$; and by substitution $\frac{(2p - x^2)^2}{16x} - \frac{(2p - x^2)^2}{8x} - xp + y = 0$, or

$p^2 + 3x^2p + \frac{x^4}{4} - 4xy = 0$ (α); which satisfies the proposed, and is a singular solution. In this case we cannot obtain a singular solution of the second kind; because $u=0$ being linear with respect to p, we cannot eliminate p by means of $\frac{du}{dp} = 0$.

It is worthy of remark that $y = -\frac{x^3}{2}$, though a singular solution of (α), will not satisfy the proposed example. It may be called, by way of contradistinction, a *double* singular solution.

Lagrange has shown that this is a property which double singular solutions, in general, possess.

The complete primitive is $y - \frac{1}{2}ax^2 - bx - ab = 0 = v$. If we eliminate both a and b by means of $\frac{dv}{da} = 0$ and $\frac{dv}{db} = 0$, we have $0 = \frac{1}{2}x^2 + b$ and $0 = x + a$; and there results $y = -\frac{x^3}{2}$, the singular solution of (α).

Ex. 15. $x\frac{d^2y^2}{dx^4} - \frac{2dy}{dx}\frac{d^2y}{dx^2} + x = 0$, or $xq^2 - 2pq + x = 0 = u$.

Here $\frac{du}{dq} = 0 = 2xq - 2p$, or $q = \frac{p}{x}$, which substituted in $u = 0$, gives $-\frac{p^2}{x} + x = 0$; or $x^2 - p^2 = 0$ is a singular solution.

The complete primitive of this example is $\frac{x^3}{3} - 2ay + a^2x + b = 0$, and the two first primitives are $x^2 - 2ap + a^2 = 0 = v$ (1); and, eliminating a,

$4x^2(y-xp)^2 - 4\left(b - \frac{2x^3}{3}\right)(y-xp)p + \left(b - \frac{2x^3}{3}\right)^2 \cdot \cdot \cdot = 0 = w$ (2).

From (1), we have $\frac{dv}{da} = 0 = -2p + 2a$, or $a = p$, which substituted in $v = 0$, gives $x^2 - p^2 = 0$.

From (2) $\frac{dw}{db} = 0 = -4(y - xp)p + 2\left(b - \frac{2x^3}{3}\right)$, or $b - \frac{2x^3}{3} = 2(y - xp)p$, which substituted in $w = 0$, gives $4x^2(y - xp)^2 - 4(y - xp)^2p^2 = 0$, or $(x^2 - p^2)(y - xp)^2 = 0$. Of these, $y - xp = 0$ is not a solution; for it gives, when differentiated, $xq = 0$, or $q = 0$. The other factor gives $x^2 - p^2 = 0$.

It appears then in this example that the two first primitives each give the same singular solution. Lagrange has shown that this is a general property, and that it belongs to equations of any order whatever. The same writer has also given a rule by which the singular solution may be obtained from the complete primitive without knowing either of the two first primitives. (Calcul des Fonct. p. 190.)

26. PRAXIS.

1. $y = x + (a - 1)^2x^{\frac{1}{2}}$; $y = a$ is a particular solution.
2. $y = x + (a - 1)^2(a - x)^2$; $a = 1$ and $a = x$ give particular solutions; and $a = \frac{x+1}{2}$ is a singular solution.
3. $xdy^2 - ydxdy + adx^2 = 0$. The complete primitive is $y = cx + \frac{a}{c}$; and $y^2 = 4ax$ is a singular solution.
4. $(b - x)dy^2 + (y - x)dxdy + ydx^2 = 0$; $(x + y)^2 = 4by$ is a singular solution.
5. $dy^3 - dx^3 = 0$; $y = x$ is a particular solution.
6. $ady - adx = \sqrt{y^2 - x^2}dx$; $y = x$ is a singular solution.
7. $a^2dy - a^2dx = (y^2 - x^2)dx$; $y = x$ is a particular solution, where $c = \infty$.
8. $dy^2 + 3x^2dydx = \left(4xy - \frac{x^4}{4}\right)dx^2$; $y = -\frac{x^3}{2}$ is a singular solution.
9. $dy = \sqrt{y - a}dx$; $y = a$ is a singular solution.

10. $dy = (2x \pm \sqrt{x^2 - y})dx$; $y = x^2$ is a singular solution.

11. $aydx = (ax - \sqrt{x^2 + y^2})dy$; $x^2 + y^2 = 0$ is a singular solution.

12. $ydy^2 + xdxdy = bdx^2$; $y = -\frac{x^2}{4b}$ is not a solution.

13. $ydy^2 - bdxdy = adx^2$; $y = -\frac{b^2}{4a}$ is not a solution.

14. $a^2dy^2 - xydxdy = x^2dx^2$; $y^2 + 4a^2 = 0$ is not a solution.

15. $ydx = x\sqrt{dx^2 + dy^2}$; $x = 0$ is a particular solution.

16. $xdx + ady = a\sqrt{dx^2 + dy^2}$. What kind of solution is $x = 0$?

17. $dx^2 = x(dx^2 + dy^2)$; $x = 0$ is a singular solution.

18. $adxdy^3 = y(dx^2 + dy^2)^2$. What kind of solution is $y = 0$?

19. $\sqrt{2xy}\sqrt{dx^2 + dy^2} = xdy + ydx$; $y = x$ is a particular solution. (Art. 3, ex. 2.)

20. $dy = \frac{e^{\frac{x}{a}}dx}{e^{\frac{x}{a}} - e}$; $x = a$ is a particular solution.

21. $xdy = \frac{dx}{lx - la}$; $x = a$ is a particular solution where $c = \infty$.

22. $\left\{y - x\frac{dy}{dx}\right\}\left\{y - x\frac{dy}{dx} + 2a\frac{dy}{dx}\right\} = b^2$.

The complete primitive is $ay = c(x - a) + a\sqrt{b^2 + c^2}$; and the singular solution is $y = \frac{b}{a}\sqrt{2ax - x^2}$.

23. $\frac{b^2}{4x^2}\left(\frac{dy}{dx} - a\right)^2 - \frac{x}{2}\left(\frac{dy}{dx} - a\right) + y - ax = 0$.

The complete primitive is

$y - ax - \frac{x^2(c-a)^2}{2} + \frac{b^2(c-a)^4}{4} = 0$; and $y = ax$ is a particular, and $y = ax + \frac{x^4}{4b^2}$ is a singular solution.

24. $\frac{d^2y^2}{dx^4} + \left\{\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right\}^2 - \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y.$

The singular solution of the first order is

$p^2 + \left(\frac{x^3}{2} + x\right)p - \frac{x^4}{16} - (1 + x^2)y = 0$ (α), whose primitive is $l.(x + \sqrt{1+x^2}) = \sqrt{16y + 4x^2 + x^4} - x\sqrt{1+x^2} + c.$

The complete primitive is $y - \frac{ax^2}{2} - bx - a^2 - b^2 = 0 = v.$

The double singular solution is $4y + x^2 + x^4 = 0$; and it will not satisfy the proposed example. It may be deduced from the primitive by eliminating a and b by means of $v = 0$, $\frac{dv}{da} = 0$, and $\frac{dv}{db} = 0$.

25. $\left\{(x-1)\frac{d^2y}{dx^2} - \frac{dy}{dx}\right\}^2 - \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

The two first primitives are

$\left\{p - (x-1)b\right\}^2 - \frac{bx^2}{2} = y$ (1), and $\frac{p^2}{x^2} + \frac{xp}{2} + \frac{ax}{2} \cdots + \frac{2a(x-1)p}{x^2} + \frac{a^2(x-1)^2}{x^2} = y$ (2). They each give the same singular solution, viz. $(x-1)(x-2)\frac{xp}{2} - \frac{x^4}{16} = (x-1)^2y$; a linear equation, whose primitive is

$y = \frac{x(x-2)}{8}\left\{c - \frac{4}{x-2} + l(x-1)\right\}.$

26. $(x^2 - x + 1)\frac{d^2y^2}{dx^4} - (2x-1)\frac{dy}{dx}\frac{d^2y}{dx^2} - \frac{x^2}{2}\frac{d^2y}{dx^2} \cdots + \frac{dy^2}{dx^2} + x\frac{dy}{dx} = y.$

The singular solution of the first order is

$p^2 + \frac{(2x^3 - 3x^2 + 4x)p}{3} = \frac{x^4}{12} + \frac{4(x^2 - x + 1)y}{3}$ (α), and the complete primitive is $y = \frac{1}{2}ax^2 + bx + a^2 + ab + b^2$.

27. $(xy - 1)\left\{2xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} + x\frac{dy^2}{dx^2}\right\}^2 \cdots - xy\left\{y + x\frac{dy}{dx}\right\}\frac{dy^2}{dx^2} = 0.$

A first primitive is $yp^2 - 2axyp + a^2x = 0$; and a singular solution is $xy = 1$.

28. $y - xp + \frac{x^2q}{2} - \frac{x^3r}{6} - p^2 + 2xpq - x^2q^2 - x^2pr +$

$x^3qr - \frac{x^4r^2}{4} = 0 = u.$

The singular solution of the second order is

$y = \frac{2xp}{3} - \frac{x^2q}{6} - \frac{x^2}{36}$, whose primitive is

$y = cx^3 + c'x^2 - \frac{x^2}{6}(1 + lx)$ (α).

This example does not admit of a singular solution of a lower order; since the equations $\frac{du}{dp} = 0$, $\frac{du}{dy} = 0$ and $\frac{du}{dr} = 0$ are such that they cannot be simultaneous.

The complete primitive is $y = \frac{ax^2}{2} - \frac{bx^3}{6} - cx - c^2$.

CHAPTER VI.

Fluxional Equations of two Variables of a higher order than the first.

1. THE general form of an equation between two variables of the nth order is $\text{F}(x, y, p, q, \cdots s, t) = 0$, where $p, q, \cdots s, t$ respectively $= \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^ny}{dx^n}$.

It has been shown, ch. 3, 13, that every equation of this form has a complete primitive containing n arbitrary constants; but we have not the means of finding this primitive except in certain cases. We cannot even integrate the general equation of the second order $\text{F}(x, y, p, q) = 0$, though all the combinations except those which contain both the variables x and y, admit of easy solutions; or at least are reducible to a simpler form.

We shall give these integrations in the next article, and shall conclude the chapter with such other forms as are suited to the nature of an elementary treatise.

In the following articles the fluxional coefficient of the highest order, viz. q, is in general supposed to be of the first degree.

2. *Required to integrate the forms,*
(1) $\text{F}(x, q) = 0$; (2) $\text{F}(y, q) = 0$; (3) $\text{F}(p, q) = 0$;
(4) $\text{F}(x, p, q) = 0$; and (5) $\text{F}(y, p, q) = 0$.

(1) $\text{F}(x, q) = 0$.

Here $\frac{d^2y}{dx^2} = fx = \text{x}$, wherefore $\frac{d^2y}{dx} = \text{x}dx$; and integrating,

$\frac{dy}{dx} = \int \text{x}dx + c$, or $y = \int dx \int \text{x}dx + cx + c'$.

To decompose this into single fluents, since
$\int dx \int \text{x}dx = x \int \text{x}dx - \int x\text{x}dx$, we also have
$y = x \int \text{x}dx - \int \text{x}xdx + cx + c'$.

Ex. $d^2y = ax^n dx^2$.

Here $\frac{d^2y}{dx} = ax^n dx \therefore \frac{dy}{dx} = \frac{ax^{n+1}}{n+1} + c \therefore$

$$y = \frac{ax^{n+2}}{(n+1)(n+2)} + cx + c'.$$

Cor. 1. If $n = -1$, $\frac{d^2y}{dx} = \frac{adx}{x} \therefore dy = alx.dx + cdx$ and

$$axlx - ax + cx + c' = axlx + cx + c'.$$

Cor. 2. If $n = -2$, $\frac{d^2y}{dx} = \frac{adx}{x^2} \therefore dy = -\frac{a}{x}dx + cdx$ and $y = -alx + cx + c'$.

(2) $\mathrm{F}(y, q) = 0$.

Here $\frac{d^2y}{dx^2} = fy = \mathrm{Y}$; wherefore, multiplying by $2dy$ and integrating, there results $\frac{dy^2}{dx^2} = 2\int \mathrm{Y}dy = \mathrm{Y}' + c \therefore \ldots$

$\frac{dy}{dx} = \sqrt{\mathrm{Y}' + c} \therefore x = \int \frac{dy}{\sqrt{\mathrm{Y}' + c}} + c'$.

Ex. 1. $a^2d^2y + ydx^2 = 0$.

Multiplying by $2dy$, $2a^2dyd^2y + 2ydydx^2 = 0$; and integrating, $a^2dy^2 + y^2dx^2 = c^2dx^2$, or $dx = \frac{ady}{\sqrt{c^2 - y^2}} \therefore \ldots$

$x = a\mathrm{sin.}^{-1}\frac{y}{c} + c'$, or $y = c\mathrm{sin.}\left(\frac{x}{a} + c'\right)$; which may be made to take the form of $y = c\mathrm{sin.}\frac{x}{a} + c'\mathrm{cos.}\frac{x}{a}$.

Cor. The primitive of $d^2y + \mathrm{A}ydx^2 = 0$ is $y = c\mathrm{sin.}\mathrm{A}^{\frac{1}{2}}x + c'\mathrm{cos.}\mathrm{A}^{\frac{1}{2}}x$.

Ex. 2. $a^2d^2y - ydx^2 = 0$.

Here $dx = \frac{ady}{\sqrt{y^2 + c^2}} \therefore \frac{x}{a} = l.\frac{y + \sqrt{y^2 + c^2}}{c'} \therefore \ldots$

$c'e^{\frac{x}{a}} = y + \sqrt{y^2 + c^2}$ and $y = \frac{c'e^{\frac{x}{a}}}{2} - \frac{ce^{-\frac{x}{a}}}{2c'}$; which is of the form $y = ce^{\frac{x}{a}} + c'e^{-\frac{x}{a}}$.

Ex. 3. $\frac{d^2y}{dx^2} = \frac{m}{(a-y)^2}$.

Here $\frac{2dyd^2y}{dx^2} = \frac{2mdy}{(a-y)^2}$ $\therefore \frac{dy^2}{dx^2} = \frac{2m}{a-y} + c$ and $0 = \frac{2m}{a} + c$, if from the nature of the question $dy = 0$ when $y = 0$; whence we have $\frac{dy^2}{dx^2} = \frac{2m}{a} \cdot \frac{y}{a-y}$ $\therefore dx = \sqrt{\frac{a}{2m}} \cdot \frac{\sqrt{a-y}.dy}{\sqrt{y}}$ . . .

$= \sqrt{\frac{a}{2m}} \cdot \frac{(a-y)dy}{\sqrt{ay-y^2}} = \sqrt{\frac{a}{2m}} \left\{ \frac{(\frac{1}{2}a-y)dy}{\sqrt{ay-y^2}} + \frac{\frac{1}{2}ady}{\sqrt{ay-y^2}} \right\}$

$\therefore x = \sqrt{\frac{a}{2m}} \left\{ \sqrt{ay-y^2} + \frac{1}{2}av.s.^{-1}\frac{2y}{a} \right\}$.

This solution is a particular primitive.

(3) $\mathrm{F}(p, q) = 0$.

Since $q = \frac{dp}{dx}$, we have $\mathrm{F}\left(p, \frac{dp}{dx}\right) = 0$, which is an equation of the first order between p and x. Solving this equation with respect to $\frac{dp}{dx}$, let $dx = \frac{dp}{\mathrm{P}}$; then $x = \mathrm{F}p + c$.

Also $dy = pdx = \frac{pdp}{\mathrm{P}}$; wherefore $y = fp + c'$; and eliminating p by means of these equations, there will result the required primitive.

Ex. 1. $d^2y = dx\sqrt{dx^2 + dy^2}$.

Here $\frac{dp}{dx} = \sqrt{1+p^2}$ $\therefore dx = \frac{dp}{\sqrt{1+p^2}}$ $\therefore$ . . . . . .

$e^x = \frac{p + \sqrt{1+p^2}}{c}$ $\therefore p = \frac{ce^x}{2} - \frac{1}{2ce^x}$ $\therefore 2y = ce^x + \frac{1}{ce^x} + c'$.

Ex. 2. $\frac{d^2y}{dx^2} + \frac{mdy^2}{dx^2} + a = 0$.

Here $\frac{dp}{dx} + mp^2 = -a$, or $dx = -\frac{dp}{mp^2+a}$; $\therefore$ . . .

$$x = -\frac{1}{\sqrt{ma}}\text{tan.}^{-1}\frac{m^{\frac{1}{2}}p}{a^{\frac{1}{2}}} + c$$

and let $0 = -\frac{1}{\sqrt{ma}} \cdot \frac{\pi}{2} + c$ $\Bigg\}$ $\therefore$ . . . . . .

$$x = \frac{1}{\sqrt{ma}}\text{cotan.}^{-1}\frac{m^{\frac{1}{2}}p}{a^{\frac{1}{2}}} \therefore p = \sqrt{\frac{a}{m}}\,\text{cotan.}\,\sqrt{ma}x \quad . \quad . \quad .$$

$$\therefore dy = \frac{1}{m}\,\text{cotan.}\,\sqrt{ma}x \times \sqrt{ma}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\therefore \quad y = \frac{1}{m}\,l.\,\text{sin.}\,\sqrt{ma}x + c.$$

Ex. 3. Required the curve whose radius of curvature is constant. (Vol. i. ch. 11, 5, cor. 1.)

The equation is $(1 + p^2)^{\frac{3}{2}} = -aq = -\frac{adp}{dx} \therefore$ . . . .

$$dx = -\frac{adp}{(1+p^2)^{\frac{3}{2}}} \therefore \text{(vol. i. ch. 2, 13, ex. 2)}\ x = c - \frac{ap}{\sqrt{1+p^2}}.$$

Also $dy = pdx = -\frac{apdp}{(1+p^2)^{\frac{3}{2}}} \therefore y = \frac{a}{\sqrt{1+p^2}} + c'$; and eliminating p, the primitive is $a^2 = (c - x)^2 + (y - c')^2$, which is a circle whose radius $= a$.

The two first primitives are

$x = c - \frac{ap}{\sqrt{1+p^2}}$ and $y = \frac{a}{\sqrt{1+p^2}} + c'$, which will each satisfy the proposed equation. They may be integrated by solving them with respect to p, which will give dy in terms of x and dx; and the result in either case is a circle.

Ex. 4. $\frac{d^2y}{dx^2} + \text{A}\frac{dy^2}{dx^2} + \text{B}\frac{dy}{dx} + \text{C} = 0.$

Here $\frac{dp}{dx} + \text{A}p^2 + \text{B}p + \text{C} = 0$, or $-\text{A}dx = \frac{dp}{p^2 + \frac{\text{B}}{\text{A}}p + \frac{\text{C}}{\text{A}}}$.

Let $-a$ and $-b$ be the roots of $p^2 + \frac{\text{B}}{\text{A}}p + \frac{\text{C}}{\text{A}} = 0$; then

$\text{A}dx = \frac{1}{a-b}\left\{\frac{dp}{p+a} - \frac{dp}{p+b}\right\}$; and integrating,

$\frac{p+a}{p+b} = ce^{\text{A}(a-b)x} \therefore p = \frac{a-bce^{\text{A}(a-b)x}}{ce^{\text{A}(a-b)x}-1} = \frac{a-b}{ce^{\text{A}(a-b)x}-1} - b \cdot \cdot \cdot$

$\therefore y = (a-b)\int \frac{dx}{ce^{\text{A}(a-b)x}-1} - bx = \frac{1}{\text{A}}\, l\, \frac{ce^{\text{A}(a-b)x}-1}{c'e^{\text{A}(a-b)x}} - bx.$

If the roots of $\text{A}p^2 + \text{B}p + \text{C} = 0$ are impossible, y is a circular function of x.

(4) $\text{F}(x, p, q) = 0$.

Since $q = \frac{dp}{dx}$, we have $\text{F}\left(x, p, \frac{dp}{dx}\right) = 0$, an equation of the first order between p and x. Suppose that it can be integrated; and let its complete primitive be $f(x, p, c) = 0$ (α).

Case 1. Suppose that (α) can be solved with respect to p, or that we have $p = \text{X}$; then dy, which $= pdx$, $= \text{X}dx$, and $y = \int \text{X}dx + c'$.

Case 2. Suppose that it can be solved with respect to x, or that $x = \text{P}$; then, since $dy = pdx$, integrating by parts, $y = px - \int \text{P}dp$; from which equation combined with (α) p may be eliminated, and we shall have the required equation between x and y.

Case 3. If (α) cannot be solved with respect to either x or p; we must look out for some new function v, so that $x = \text{V}$, $p = \text{W}$, where V and W are each functions of v; then, we have $y = \int pdx = \int \text{W}d\text{V}$, from which v is to be eliminated by means of $x = \text{V}$.

Ex. 1. $dxdy - xd^2y = xdy^2$.

Here $p - \frac{xdp}{dx} = xp^2 \therefore \frac{pdx - xdp}{p^2} = xdx \therefore$, integrating,

$\frac{x}{p} = \frac{x^2-c^2}{2}$, or $dy = \frac{2xdx}{x^2-c^2} \therefore y + c' = l(x^2 - c^2)$, or

$c'e^y = x^2 - c^2$.

Ex. 2. $2x(dx^2 + dy^2)^{\frac{3}{2}} = a^2dxd^2y$.

Here $(1 + p^2)^{\frac{3}{2}} = \frac{a^2q}{2x} = \frac{a^2dp}{2xdx}$, or $2xdx = \frac{a^2dp}{(1+p^2)^{\frac{3}{2}}} \cdot \cdot \cdot$

$\therefore x^2 + c^2 = \frac{a^2p}{\sqrt{1+p^2}} \therefore p = \frac{x^2+c^2}{\sqrt{a^4-(x^2+c^2)^2}} \cdot \cdot \cdot \cdot \cdot$

$\therefore y = \int \frac{(x^2+c^2)dx}{\sqrt{a^4-(x^2+c^2)^2}} = (\text{if } c=0)\int \frac{x^2dx}{\sqrt{a^4-x^4}}$, which may be computed between the values $x=0$, $x=a$. (Ch. 2, 8, ex. 2.)

This is the equation of the rectangular elastick curve whose radius of curvature varies inversely as the abscissa. By the same process, the curve may be found whose radius of curvature varies according to any known function of the abscissa.

Ex. 3. $dxdy^2 + dx^3 + xdyd^2y = a\sqrt{dy^2+dx^2}.d^2y$.

Substituting $q = \frac{dp}{dx}$, and dividing by $\sqrt{1+p^2}$, we have

$\sqrt{1+p^2}dx + \frac{xpdp}{\sqrt{1+p^2}} = adp$; and, integrating, ...

$x\sqrt{1+p^2} = ap + c$; which solved with respect to p, and integrated, will give the primitive.

Or; since $y = px - \int xdp = px - \int \frac{(ap+c)dp}{\sqrt{1+p^2}}$, we have

$y = px - a\sqrt{1+p^2} - cl\frac{p+\sqrt{1+p^2}}{c'}$ . . . . . .

$= \frac{cp-a}{\sqrt{1+p^2}} - cl\frac{p+\sqrt{1+p^2}}{c'}$; from which p may be eliminated by means of $x = \frac{ap+c}{\sqrt{1+p^2}}$, and there results

$y = \sqrt{a^2+c^2-x^2} - cl\frac{c+\sqrt{a^2+c^2-x^2}}{c'(x-a)}$.

Ex. 4. $2(a^2dy^2 + x^2dx^2)d^2y = xdydx^3$.

Here $2(a^2p^2+x^2)dp = xpdx$ is an homogeneous equation; substitute therefore $x=pz$; and dividing by p^2, there results

$2(a^2+z^2)dp = z(pdz + zdp)$, or $\frac{dp}{p} = \frac{zdz}{2a^2+z^2}$ . . .

$\therefore p = \frac{\sqrt{2a^2+z^2}}{c}$, or $p^2 = \frac{\sqrt{2a^2p^2+x^2}}{c}$.

To obtain the primitive from this equation, we have

$p^4 - \frac{2a^2p^2}{c^2} = \frac{x^2}{c^2}$ $\therefore p = \frac{\sqrt{a^2+\sqrt{a^4+c^2x^2}}}{c}$, or

$dy = \frac{\sqrt{a^2+\sqrt{a^4+c^2x^2}}}{c}dx = \frac{xdx}{\sqrt{\sqrt{a^4+c^2x^2}-a^2}}$ =, by substitution, $\frac{2}{c^2}\frac{v^3dv}{\sqrt{v^2-a^2}}$ $\therefore y = \frac{2}{3c^2}\sqrt{v^2-a^2}(v^2+2a^2)+c'$... (vol. i. ch. 2, 53, ex. 3) where $v^2 = \sqrt{a^4+c^2x^2}$.

Ex. 5. $m(1+p^2)^{\frac{3}{2}} + p(1+p^2) + qx = 0$, or

$$\frac{qx}{(1+p^2)^{\frac{3}{2}}} + \frac{p}{\sqrt{1+p^2}} + m = 0.$$

Substituting $q = \frac{dp}{dx}$, we have

$$\frac{xdp}{(1+p^2)^{\frac{3}{2}}} + \frac{pdx}{\sqrt{1+p^2}} + mdx = 0;$$ and, integrating,

$$\frac{xp}{\sqrt{1+p^2}} + mx = c \therefore \frac{x^2p^2}{1+p^2} = c^2 - 2mcx + m^2x^2 \cdots$$

$$\therefore \frac{x^2}{1+p^2} = (1-m^2)x^2 + 2mcx - c^2 \cdots$$

$$\therefore 1+p^2 = \frac{x^2}{(1-m^2)x^2+2mcx-c^2} \cdots$$

$$\therefore p^2 = \frac{m^2x^2-2mcx+c^2}{(1-m^2)x^2+2mcx-c^2} \cdots$$

$$\therefore p = \frac{mx-c}{\sqrt{(1-m^2)x^2+2mcx-c^2}} \cdots$$

$$= \frac{1}{\sqrt{1-m^2}} \cdot \frac{mx-c}{\sqrt{x^2+\frac{2mc}{1-m^2}x-\frac{c^2}{1-m^2}}};$$ which

may be integrated by substituting $z = x + \frac{mc}{1-m^2}$.

(5) $\mathrm{F}(y, p, q) = 0$.

Let the equation be reduced to the form $q = f(y, p)$; then, since $dy = pdx$ and $dp = qdx$, we have, eliminating dx, $\frac{dy}{p} = \frac{dp}{q}$, or $q = \frac{pdp}{dy}$.

Substitute this value of q in $q = f(y, p)$; and there results $pdp = f(y, p)dy$, which is an equation of the first

order between y and p. Suppose that it can be integrated; and let its primitive be $\varphi(y, p, c) = 0$ (α).

Case 1. Let $p = \text{Y}$; then $dx = \frac{dy}{\text{Y}}$ and $x = \int \frac{dy}{\text{Y}}$.

Case 2. Let $y = \text{P}$; then p, which $= \frac{dy}{dx}$, $= \frac{d\text{P}}{dx}$; wherefore $dx = \frac{d\text{P}}{p}$ and $x = \int \frac{d\text{P}}{p}$, from which p may be eliminated by means of $y = \text{P}$.

Case 3. If (α) cannot be solved with respect to either y or p, we must find some third function v, so that $y = \text{v}$, $p = \text{w}$; then, since $dy = pdx$, we have $d\text{v} = \text{w}dx$ and $x = \int \frac{d\text{v}}{\text{w}}$; from which v is to be eliminated by means of $y = \text{v}$.

Ex. 1. $yd^2y + dy^2 + dx^2 = 0$.

Here q or $\frac{pdp}{dy} = -\frac{1+p^2}{y}$, $\therefore \frac{dy}{y} = -\frac{pdp}{1+p^2}$ . . .

$\therefore y = \frac{c}{\sqrt{1+p^2}}$, which is one of the first primitives; and for the other, we have $\text{P} = \frac{c}{\sqrt{1+p^2}}$ $\therefore dx$, which $= \frac{d\text{P}}{p}$, $= \frac{-cdp}{(1+p^2)^{\frac{3}{2}}}$ $\therefore x = c' - \frac{cp}{\sqrt{1+p^2}}$.

To obtain the complete primitive, we may either eliminate p from the two first primitives, or by means of $y = \frac{c}{\sqrt{1+p^2}}$, we have

$p = \frac{\sqrt{c^2-y^2}}{y}$, or $dx = \frac{ydy}{\sqrt{c^2-y^2}}$ and $x = c' - \sqrt{c^2-y^2}$, which is of the form $x^2 + 2c'x + y^2 = c^2$.

Ex. 2. $(ydy + adx)d^2y = dy(dx^2 + dy^2)$.

Substituting $q = \frac{pdp}{dy}$, there will result

$dp(yp + a) = dy(1 + p^2)$, or $dy - \frac{ypdp}{1+p^2} = \frac{adp}{1+p^2}$ a linear equation $\therefore$ (ch. 4, 18, ex. 3) $y = ap + c\sqrt{1+p^2}$ . . .

$\therefore dx = \frac{dy}{p} = \frac{adp}{p} + \frac{cdp}{\sqrt{1+p^2}}$; which being integrated, and p eliminated by means of the value of y, will give the required primitive.

3. *Certain equations of the form* $\mathrm{F}(x, y, p, q) = 0$, *which are to be integrated by particular artifices of calculation.*

Ex. 1. $adx = \frac{xyd^2y + xdy^2}{dx}$.

Here $\frac{adx}{x} = \frac{yd^2y + dy^2}{dx}$ $\therefore$ integrating, $alx = \frac{ydy}{dx} + c$ $\therefore$ $alxdx = ydy + cdx$, and integrating, $\frac{y^2}{2} + cx + c'$ . . . $= afl xdx = alx \,.\, x - ax$; or changing the constant, $\frac{y^2}{2} + cx + c' = axlx$.

4. *Homogeneous equations of the second order are reducible to the first order.*

Equations of the second order are said to be homogeneous when they are so with respect to d^2y, dy, dx, y and x. Or, expressed in terms of p and q, if they are homogeneous, the dimensions of p not being taken into the account, and q considered as of -1 dimensions.

Assume $y = vx$, $q = \frac{w}{x}$ } which substituted in the proposed, dividing by x, will give an equation between w, v and p. Let it be $u = 0$ (α).

Since dy or $pdx = vdx + xdv$, we have $\frac{dx}{x} = \frac{dv}{p-v}$. Also dp or $qdx = \frac{wdx}{x}$, or $\frac{dx}{x} = \frac{dp}{w}$; wherefore $\frac{dp}{w} = \frac{dv}{p-v}$ (β).

Substitute in this expression the value of w deduced from $u = 0$, and there will result an equation of the first order between p and v. If this can be integrated, so that p may be expressed in terms of v, we shall have from the equation $\frac{dx}{x} = \frac{dv}{p-v}$, x in terms of v; and consequently y, which $= vx$, may be found.

Cor. 1. The proposed equation is always integrable when (β) is integrable; and the result is the complete fluent, because the double integration will introduce two arbitrary constants.

Cor. 2. The proposed is integrable when w is a sole function of p; for then (β) is linear with respect to v.

Cor. 3. The proposed is also integrable when w is a function of $p - v$.

For substitute $s = p - v$; then $p = s + v$ and (β) becomes $sds + sdv = sdv \therefore v = \int \frac{sds}{s - s} \therefore$ &c.

Ex. 1. $x^2d^2y = xdxdy + 3ydx^2$.

Assume $y = vx$, $q = \frac{w}{x}$ $\Big\} \therefore xw = xp + 3vx$, or $w = p + 3v$;

wherefore by substitution, $\frac{dp}{p+3v} = \frac{dv}{p-v}$ . . . . . or $(p - v)dp = (p + 3v)dv$, or $pdp = vdp + pdv + 3vdv \therefore p^2 = 2vp + 3v^2 + c^2$ and $p = v + \sqrt{4v^2 + c^2} \therefore$ by substitution $\frac{dx}{x} = \frac{dv}{p-v} = \frac{dv}{\sqrt{4v^2+c^2}} \therefore$ integrating,

$x^2 = \frac{2v + \sqrt{4v^2+c^2}}{c'} = \frac{v + \sqrt{v^2+c}}{c'} = \frac{y + \sqrt{y^2+cx^2}}{c'x}$; and by reduction, $y = \frac{c'^2x^6 - cx^2}{2c'x^3} = cx^3 + \frac{c'}{x}$.

Ex. 2. $(dx^2 + dy^2)^{\frac{3}{2}} = ndxd^2y\sqrt{x^2 + y^2}$.

By substitution, $(1 + p^2)^{\frac{3}{2}} = nw\sqrt{1 + v^2}$ . . . . . $\therefore w = \frac{(1+p^2)^{\frac{3}{2}}}{n\sqrt{1+v^2}}$; and by substitution, $\frac{n\sqrt{1+v^2}}{(1+p^2)^{\frac{3}{2}}}dp = \frac{dv}{p-v}$ or $\frac{n(p-v)dp}{(1+p^2)^{\frac{3}{2}}} = \frac{dv}{\sqrt{1+v^2}}$; which is integrated by means of circular arcs by Euler and by Garnier. Calc. Int. p. 234.

Ex. 3. $3ydx^2 - 3xdxdy + 4y^2d^2y = 0$.

By substitution $3v - 3p + 4v^2w = 0$, or $w = \frac{3(p - v)}{4v^2}$

$\therefore$ (β) becomes $\frac{4v^2dp}{3} = dv$, or $\frac{4dp}{3} = \frac{dv}{v^2}$ $\therefore$ integrating, $\cdots$ $p = c - \frac{3}{4v}$ $\therefore \frac{dx}{x}$, which $= \frac{dv}{p-v} = \frac{-4vdv}{4v^2-4cv+3}$. Let $2v + a$ and $2v + b$ be the factors of $4v^2 - 4cv + 3$; then $-\frac{dx}{x} = \frac{a}{a-b}\frac{2dv}{2v+a} - \frac{b}{a-b}\frac{2dv}{2v+b}$, and integrating,

$\frac{c'}{x} = \frac{(2v+a)^{\frac{a}{a-b}}}{(2v+b)^{\frac{b}{a-b}}}$; or $c'(2y + bx)^b = (2y + ax)^a$; which is the complete primitive, because c is contained in a and b.

Ex. 4. $nx^3d^2y = (ydx - xdy)^2$.

By substitution we have $nx^3w = (vx - xp)^2$, or $w = \frac{1}{n}(v - p)^2 = \frac{s^2}{n}$: and ($\beta$) becomes $\frac{n(dv+ds)}{s^2} = \frac{dv}{s}$, or $dv = \frac{nds}{s-n}$ $\therefore v = nl\frac{s-n}{c}$. Also $\frac{dx}{x} = \frac{dv}{p-v} = \frac{nds}{s(s-n)}$ $= \frac{ds}{s-n} - \frac{ds}{s}$, or $x = \frac{s-n}{c's}$. But $y = vx = nxl \cdot \frac{s-n}{c} \cdots$ $= nxl\frac{nc'x}{c(1-c'x)}$.

Otherwise. Since $nx^3q = (y - xp)^2$, substitute $y - xp = v$; and differentiating, we have $-xq = \frac{dv}{dx}$ $\therefore$ by substitution, $-\frac{nx^2dv}{dx} = v^2$, or $-\frac{ndv}{v^2} = \frac{dx}{x^2}$ $\therefore \frac{1}{x} = \frac{1}{c} - \frac{n}{v} = \frac{1}{c} - \frac{n}{y-xp}$, or $y - xp = \frac{ncx}{x-c}$ $\therefore \frac{ydx-xdy}{x^2} = \frac{ncdx}{x(x-c)}$; and, integrating, $\frac{y}{x} = nl \cdot \frac{x-c}{c'x}$.

5. *Equations of the second order which are homogeneous only with respect to y, dy, and d^2y are reducible to the first order.*

Assume $\left.\begin{matrix} p = vy \\ q = wy \end{matrix}\right\}$; then, since (ex hyp.) the proposed is homogeneous with respect to y, p, and q; the same power of y

will be found in each term of the resulting equation; which therefore by division becomes an equation between v, w, and x. Let it be $u = 0$ (α).

Since $dy = pdx = vydx$, we have $\frac{dy}{y} = vdx$. Also dp, which $= qdx = wydx, = vdy + ydv$; wherefore $\frac{dy}{y} = \frac{wdx - dv}{v}$, or $vdx = \frac{wdx - dv}{v}$, or $dv + v^2dx = wdx$ (β).

Substitute in (β) the value of w deduced from (α), and there will result an equation which when integrated will give v in terms of x; and to find y in terms of x, we have $\frac{dy}{y} = vdx$, or $y = ce^{\int vdx}$.

Ex. 1. $yd^2y - dy^2 = xydxdy$, or $yq - p^2 = xyp$.

Substituting $p = vy$, $q = wy$; and dividing by y^2, there results $w - v^2 = xv$, or $w = xv + v^2$ (α).

Substituting this value of w in (β) it becomes

$dv + v^2dx = (xv + v^2)dx$ or $\frac{dv}{v} = xdx \therefore v = ce^{\int xdx}$ . . .

$\therefore \frac{dy}{y} = ce^{\int xdx}dx$ and $y = c'e^{ce^{\int xdx}dx}$.

Ex. 2. The linear equation $d^2y + Pdydx + Qydx^2 = 0$, where P and Q are sole functions of x.

Here $q + Pp + Qy = 0$; which by substitution and division becomes $w + Pv + Q = 0$, or $w = -Pv - Q$; and (β) becomes $dv + (v^2 + Pv + Q)dx = 0$; from which v may be obtained in terms of x, and consequently y, which $= ce^{\int vdx}$, may be found.

Ex. 3. Required the spiral whose curvature varies inversely as the radius vector.

Here $\frac{(y^2 + p^2)^{\frac{3}{2}}}{y^2 + 2p^2 - yq} = ny$, where the angle described by the radius vector y is the independent variable. (Vol. i. ch. 11, 10.)

Since this equation is homogeneous with respect to y, p and q, substitute $p = zy$; then, since $q = \frac{pdp}{dy}$, we have, dividing by y^3, $(1 + z^2)^{\frac{3}{2}} = n + 2nz^2 - \frac{nz(zdy + ydz)}{dy}$ . . .

$= n + nz^2 - \frac{nyzdz}{dy}$ ∴ . . . . . .

$\frac{dy}{y} = \frac{-nzdz}{(1+z^2)^{\frac{3}{2}} - n(1+z^2)} = \frac{nzdz}{(1+z^2)\{n - \sqrt{1+z^2}\}}$

Also $dx = \frac{dy}{p} = \frac{dy}{zy} = \frac{ndz}{(1+z^2)\{n - \sqrt{1+z^2}\}}$.

Cor. If $n = 1$, there will result

$$x = c' + \sqrt{\frac{2y-c}{c}} + \cos.^{-1}\frac{y-c}{y}.$$

Or if $\varphi = \cos.^{-1}\frac{y-c}{y}$, we shall have $y = \frac{c}{1-\cos.\varphi}$ and

$x = \alpha + \varphi + \cot.\frac{\varphi}{2}$.

Ex. 4. $dy^2 - yd^2y = n\sqrt{dy^2dx^2 + a^2d^2y^2}$, or . . . .

$p^2 - yq = n\sqrt{p^2 + a^2q^2}$.

Substitute $q = zp$; then the proposed becomes

$p - yz = n\sqrt{1 + a^2z^2}$.

Differentiating this, since $dp = qdx = zpdx = zdy$, we have

$$-ydz = \frac{na^2zdz}{\sqrt{1+a^2z^2}}, \text{ or } \left\{\frac{na^2z}{\sqrt{1+a^2z^2}} + y\right\}dz = 0.$$

This is satisfied either by $dz = 0$, an equation of the third order, which belongs to the complete primitive; or by $\frac{na^2z}{\sqrt{1+a^2z^2}} + y = 0$, an equation of the second order, from which the singular solution is to be obtained (ch. 5, 17).

The first gives $z = c$ or $q = cp$; and by substitution in the proposed, there results $p - cy = n\sqrt{1 + a^2c^2}$; and integrating, $c'e^{ax} = cy + n\sqrt{1 + a^2c^2}$.

To find the singular solution, we have $\frac{na^2z}{\sqrt{1+a^2z^2}} + y = 0$; and consequently p, which $= yz + n\sqrt{1 + a^2z^2}$, . . .

$= n\sqrt{1 + a^2z^2} - \frac{na^2z^2}{\sqrt{1+a^2z^2}} = \frac{n}{\sqrt{1+a^2z^2}}$ and $dx = \frac{dy}{p}$ . . .

$= -\frac{na^2dz}{(1+a^2z^2)^{\frac{3}{2}}} \div \frac{n}{\sqrt{1+a^2z^2}} = -\frac{a^2dz}{1+a^2z^2}$, or . . . .

$x = c - a\tan.^{-1}az$; from which eliminating az by means of $\frac{na^2z}{\sqrt{1+a^2z^2}} + y = 0$, there results $\frac{c-x}{a}$ . . . . . $= \tan.^{-1}\frac{y}{\sqrt{n^2a^2-y^2}} = \sin.^{-1}\frac{y}{na}$, or $y = na\sin.\frac{c-x}{a}$.

This is the same solution as that which results from eliminating q by means of $(p^2 - yq)^2 - n^2(p^2 + a^2q^2) = 0 = u$ and $\frac{du}{dq} = 0$.

There is a double singular solution, viz. $y = na$.

Ex. 5. $myd^2y + ndy^2 = \frac{ydxdy}{\sqrt{a^2+x^2}}$, or . . . . . $myq + np^2 = \frac{yp}{\sqrt{a^2+x^2}}$.

Substituting $p = vy$, $q = wy$, we have $mw + nv^2 = \frac{v}{\sqrt{a^2+x^2}}$, or $w = \frac{v}{m\sqrt{a^2+x^2}} - \frac{nv^2}{m}$ $\therefore$ (β) becomes $dv + v^2dx = \frac{vdx}{m\sqrt{a^2+x^2}} - \frac{nv^2dx}{m}$, or . . . . . $dv + \frac{m+n}{m}v^2dx = \frac{vdx}{m\sqrt{a^2+x^2}}$, or $-\frac{dv}{v^2} + \frac{1}{v} \cdot \frac{dx}{m\sqrt{a^2+x^2}} = \frac{m+n}{m}dx$ $\therefore$ integrating,

$$\frac{1}{v} = \frac{1}{(x+\sqrt{a^2+x^2})^{\frac{1}{m}}}\left\{c + \frac{m+n}{m}\int(x + \sqrt{a^2+x^2})^{\frac{1}{m}}dx\right\}.$$

Substitute $z = (x+\sqrt{a^2+x^2})^{\frac{1}{m}}$; then $x = \frac{z^m}{2} - \frac{a^2z^{-m}}{2}$ and $\int(x + \sqrt{a^2+x^2})^{\frac{1}{m}}dx = \frac{m}{2}\left\{\frac{z^{m+1}}{m+1} + \frac{a^2z^{1-m}}{1-m}\right\}$ $\therefore$ . . . .

$$\frac{1}{v} = \frac{c}{z} + \frac{m+n}{2}\left\{\frac{z^m}{m+1} + \frac{a^2z^{-m}}{1-m}\right\}.$$

But if $s = \frac{1}{v}$; we have $ds + \frac{sdx}{m\sqrt{a^2+x^2}} = \frac{m+n}{m}dx$, or

$$\frac{ds}{s}+\frac{-dx}{m\sqrt{a^2+x^2}}=\frac{m+n}{m}\frac{dx}{s}=\frac{m+n}{m}vdx=\frac{m+n}{m}\frac{pdx}{y}$$

$$=\frac{m+n}{m}\frac{dy}{y}\text{ .·. integrating, }c'y^{\frac{m+n}{m}}=sz\text{, or }s=\frac{1}{v}\ \cdot\ \cdot\ \cdot$$

$$=\frac{c'y^{\frac{m+n}{m}}}{z}\text{, which }=\frac{c}{z}+\frac{m+n}{2}\left\{\frac{z^m}{m+1}+\frac{a^2z^{-m}}{1-m}\right\}\text{; or the re-}$$

quired primitive is $y^{\frac{m+n}{m}}=c+c'\left\{\frac{z^{m+1}}{m+1}+\frac{a^2z^{1-m}}{1-m}\right\}$, · · ·

where $z=(x+\sqrt{a^2+x^2})^{\frac{1}{m}}$.

Otherwise. Multiply $myq+np^2=\frac{yp}{\sqrt{a^2+x^2}}$ by $\frac{dx}{yp}$; then, since $qdx=dp$ and $pdx=dy$, we have $\frac{mdp}{p}+\frac{ndy}{y}$ · · ·

$$=\frac{-dx}{\sqrt{a^2+x^2}}\text{, where the variables are separated; and con-}$$

sequently $p^my^n=c^m(x+\sqrt{a^2+x^2})$.·. $y^n dy$ · · · · ·

$$=c(x+\sqrt{a^2+x^2})^{\frac{1}{m}}dx\text{ and }y^{\frac{m+n}{m}}=c'+c\left\{\frac{z^{m+1}}{m+1}+\frac{a^2z^{1-m}}{1-m}\right\}$$

This latter method is applicable whenever the proposed equation is of the form $\mathrm{P}dp+\mathrm{Y}dy+\mathrm{X}dx=0$, which is the same as $\mathrm{P}q+\mathrm{Y}p+\mathrm{X}=0$.

The examples of this article may be integrated by substituting $y=e^{\int zdx}$; for then $p=e^{\int zdx}z$ and

$q=e^{\int zdx}\left\{z^2+\frac{dz}{dx}\right\}$; and dividing by the highest power of y, there will result an equation of the first order between z and x.

6. *Equations of the second order are integrable, which are homogeneous if* y, $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *be considered of* n, $n-1$, *and* $n-2$ *dimensions respectively.*

For, substitute $y=x^nz$, $p=x^{n-1}v$ and $q=x^{n-2}w$;

then $dy = nx^{n-1}zdx + x^n dz = x^{n-1}vdx$
$d^2y = (n-1)x^{n-2}vdx^2 + x^{n-1}dvdx = x^{n-2}wdx^2$ }.

From the first, we have $nzdx + xdz = vdx$, or

$$\frac{dx}{x} = \frac{dz}{v-nz}.$$

From the second, $(n-1)vdx + xdv = wdx$, or

$$\frac{dx}{x} = \frac{dv}{w-(n-1)v}$$; wherefore

$(w-(n-1)v)dz = (v-nz)dv$ (α).

Substitute in (α) the value of w as deduced from the proposed example, and there will result an equation of the first order between z and v; which being integrated, will give v in terms of z. Substitute this value of v in $\frac{dx}{x} = \frac{dz}{v-nz}$; and by integration, the equation between x and z, or between x and y may be found.

Ex. 1. $x^2d^2y = axdxdy + bydx^2$, or $x^2q = axp + by$.

Here the equation is homogeneous if $n = 0$; and (α) becomes $(w+v)dy = vdv$; but from the example $w = av + by$
$\therefore$ (α) is $(by + (a+1)v)dy = vdv$.

To integrate this, substitute $v = yt$; then, if $s = a + 1$, $(b + st)dy = t(ydt + tdy)$, or $-\frac{dy}{y} = \frac{tdt}{t^2-st-b}$ . . .

$$= \frac{tdt - \frac{1}{2}sdt}{t^2-st-b} + \frac{\frac{1}{2}sdt}{t^2-st-b}.$$

Let $t^2-st-b = (t-m)(t+n)$ where $s = m-n$ and $b = mn$; then, $\frac{dt}{t^2-st-b} = \frac{1}{m+n}\left\{\frac{dt}{t-m} - \frac{dt}{t+n}\right\}$ $\therefore$ integrating,

$$\frac{c}{y} = \sqrt{t^2-st-b}\left(\frac{t-m}{t+n}\right)^{\frac{m-n}{2(m+n)}}, \text{ or } \frac{c^2}{y^2} = (t-m)^{\frac{2m}{m+n}}(t+n)^{\frac{2n}{m+n}}$$
$\therefore$ $c^{m+n} = (v-my)^m(v+ny)^n$.

Since $v = px$, this is an equation of the first order and of the $m + n$th degree; and when resolved into its factors and integrated, the result is $y = c'\left\{\frac{x^m}{c^m} + \frac{c^n}{x^n}\right\}$.

Ex. 2. $x^4d^2y = (x^3 + 2xy)dxdy - 4y^2dx^2$, or
$x^4q = (x^3 + 2xy)p - 4y^2$.

This is homogeneous, if y, p and q be considered of 2, 1 and 0 dimensions respectively. Substitute $y = x^2z$; $p = xv$;

then (α) becomes $(2zv - 4z^2)dz = (v - 2z)dv$, or $(v - 2z)(2zdz - dv) = 0$.

First take $2zdz - dv = 0$ $\therefore$ integrating, $v = z^2 + c$, and by substitution, we have $\frac{dx}{x} = \frac{dz}{z^2 - 2z + c}$.

If $c = 1$, we have $lx = \frac{1}{1-z} + c'$.

If $c < 1$, let $c = 1 - c^2$; then $\frac{dx}{x} = \frac{dz}{(z-1)^2 - c^2}$ and

$$lx = -\frac{1}{2c}\, l\, \frac{c+z-1}{c-z+1} + lc'.$$

If $c > 1$, let $c = 1 + c^2$; then $\frac{dx}{x} = \frac{dz}{(z-1)^2 + c^2}$ and

$$lx = \frac{1}{c}\tan.^{-1}\frac{z-1}{c} + lc'.$$

Eliminating z from these by means of $y = x^2 z$, the three equations which include all the possible cases, and which together constitute the complete primitive are

$$lx = \frac{x^2}{x^2 - y} + c'; \quad \frac{x}{c'} = \left\{\frac{(c+1)x^2 - y}{(c-1)x^2 + y}\right\}^{\frac{1}{2c}}; \text{ and}$$

$$l\,\frac{x}{c'} = \frac{1}{c}\tan.^{-1}\frac{y - x^2}{cx^2}.$$

Next take $v - 2z = 0$; which gives $\frac{p}{x} - \frac{2y}{x^2} = 0$, or $\frac{dy}{y} - \frac{2dx}{x} = 0$, and integrating, $y = cx^2$; and since this, when differentiated, satisfies the equation, but is not included in the complete primitive, it is to be considered as a singular solution.

By means of the forms of art. 2, equations of a higher order than the second may frequently be integrated; of which we shall give instances in the following articles.

7. *Required to integrate* $\frac{d^n y}{dx^n} = \mathrm{x}$, *i. e. to find* $\int^n \mathrm{x} dx^n$.

This is to be integrated in the same manner as form (1); for

$$\int^2 \mathrm{x} dx^2 = \int dx \int \mathrm{x} dx = x\int \mathrm{x} dx - \int x\mathrm{x} dx$$

$$\int^3 x dx^3 = \int dx \int^2 x dx^2 = \int x dx \int x dx - \int dx \int x x dx$$

$$= \frac{1}{1.2}\left\{ x^2 \int x dx - 2x \int x x dx + \int x^2 x dx \right\}.$$

Similarly, $\int^4 x dx^4 = \frac{1}{1.2.3}\left\{ x^3 \int x dx - 3x^2 \int x x dx \cdot \cdot \cdot \cdot \right.$

$$\left. + \frac{3.2}{1.2} x \int x^2 x dx - \int x^3 x dx \right\}.$$

Generally it may be demonstrated by the method of induction that $y = \int^n x dx^n = \frac{1}{1.2.3 \ldots n-1}\left\{ x^{n-1} \int x dx \cdot \cdot \cdot \right.$

$$\left. - (n-1) x^{n-2} \int x x dx + \frac{(n-1)(n-2)}{1.2} x^{n-3} \int x^2 x dx \cdot \cdot \cdot \right.$$

$$\left. \pm \int x^{n-1} x dx \right\}, \text{ according as } n \text{ is odd or even.}$$

The arbitrary constants are involved in the n fluents which compose the value of y; and the part to be added is manifestly of the form $c + c'x + c''x^2 \cdots + c^{(n-1)} x^{n-1}$.

The fluent may be also developed in a series in which no integrations are necessary; for, integrating by parts, we have (vol. i. ch. 4, 46.)

$$\int x dx = x x - \int x \frac{dx}{dx} dx = x x - \frac{x^2}{2} \frac{dx}{dx} + \tfrac{1}{2} \int x^2 \frac{d^2 x}{dx^2} dx$$

$$= \frac{x x}{1} - \frac{x^2}{2} \frac{dx}{dx} + \frac{x^3}{2.3} \frac{d^2 x}{dx^2} - \frac{x^4}{2.3.4} \frac{d^3 x}{dx^3} + \cdots$$

$$\int^2 x dx^2 = \int dx \int x dx = \int dx \left\{ \frac{x x}{1} - \frac{x^2}{2} \frac{dx}{dx} + \cdots \right\}$$

$$= \frac{x x^2}{2} - \frac{x^3}{2.3} \frac{dx}{dx} + \frac{x^4}{2.3.4} \frac{d^2 x}{dx^2} - \cdots$$

$$- \frac{1}{1.2}\left\{ \frac{x^3}{3} \frac{dx}{dx} - \frac{x^4}{3.4} \frac{d^2 x}{dx^2} + \cdots \right\}$$

$$+ \frac{1}{1.2.3}\left\{ \frac{x^4}{4} \frac{d^2 x}{dx^2} - \cdots \right\}$$

$$- \frac{2.3}{2} x^4 \quad - \text{ \&c.}$$

$$= \frac{x x^2}{1.2} - \frac{2x^3}{1.2.3} \frac{dx}{dx} + \frac{}{1.2.3.4} \frac{d^2 x}{dx^2} - \cdots$$

Similarly it may be shown that

$$\int^3 \mathrm{x} dx^3 = \frac{\mathrm{x}x^3}{1.2.3} - \frac{3x^4}{1.2.3.4}\frac{d\mathrm{x}}{dx} + \frac{\frac{3.4}{2}x^5}{1.2.3.4.5}\frac{d^2\mathrm{x}}{dx^2} - \cdots;$$ and it may be demonstrated by the method of induction, that

$$\int^n \mathrm{x} dx^n = \frac{\mathrm{x}x^n}{1.2.3...n} - \frac{nx^{n+1}}{1.2...(n+1)}\frac{d\mathrm{x}}{dx} + \frac{\frac{n(n+1)}{2}x^{n+2}}{1.2...(n+2)}\frac{d^2\mathrm{x}}{dx^2} \cdots$$
$$- \frac{\frac{n(n+1)(n+2)}{2.3}x^{n+3}}{1.2...(n+3)}\frac{d^3\mathrm{x}}{dx^3} + \cdots$$ to which is to be added, as before, the part which involves the arbitrary constants, viz. $c + c'x + c''x^2 \cdots + c^{(n-1)}x^{n-1}$.

The form $\frac{d^n y}{dx^n} = \mathrm{Y}$ where Y is any function of y can be integrated only when $n = 2$ as in art. 2. (2)

Cor. The complete primitive of $\frac{d^n y}{dx^n} = 0$ is . . . . .
$y = c + c'x + c''x^2 \cdots + c^{(n-1)}x^{n-1}$.

8. *Required to integrate* $\mathrm{F}\left(\frac{d^n y}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}\right) = 0$.

This may be integrated as form (2); for, taking a particular case, let the equation be $\mathrm{F}\left(\frac{d^4 y}{dx^4}, \frac{d^2 y}{dx^2}\right) = 0$.

Substitute $q = \frac{d^2 y}{dx^2}$; and the equation becomes $\mathrm{F}\left(\frac{d^2 q}{dx^2}, q\right) = 0$; and proceeding as in form (2), we have $dx = \frac{dq}{\sqrt{\mathrm{Q}' + c}}$ and $x = \int \frac{dq}{\sqrt{\mathrm{Q}' + c}} + c'$.

Also $\frac{d^2 y}{dx} = q dx = \frac{q dq}{\sqrt{\mathrm{Q}' + c}}$ $\therefore \frac{dy}{dx} = \int \frac{q dq}{\sqrt{\mathrm{Q}' + c}} = \mathrm{Q}'' + c''$
$\therefore y = \int \mathrm{Q}'' dx + \mathrm{C}''x + c''' = \mathrm{Q}''' + \mathrm{C}''x + c'''$; and eliminating q from these values of x and y, there will result the complete primitive containing four arbitrary constants.

Ex. $\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2}$.

Here $\frac{d^2q}{dx^2} = q$; wherefore multiplying by $2dq$, and integrating, $\frac{dq^2}{dx^2} = q^2 + c^2 \therefore dx = \frac{dq}{\sqrt{q^2 + c^2}}$ and $x = l\frac{q + \sqrt{q^2+c^2}}{c'}$.

Also $\frac{d^2y}{dx} = qdx = \frac{qdq}{\sqrt{q^2 + c^2}} \therefore \frac{dy}{dx} = \sqrt{q^2 + c^2} + c'' \cdots$ $\therefore dy = dq + c''dx \therefore y = q + c''x + c'''$.

To eliminate q, we have $c'e^x = q + \sqrt{q^2 + c^2} \therefore$ . . . $c'^2e^{2x} - 2qc'e^x = c^2 \therefore q = \frac{c'^2e^{2x} - c^2}{2c'e^x} = \frac{c'e^x}{2} - \frac{ce^{-x}}{2c'}$, or the required primitive is under the form $y = ce^x + c'e^{-x} + c''x + c'''$.

9. *Required to integrate* $\text{F}\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}\right) = 0$.

Thus, let $\text{F}\left(\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}\right) = 0$, which is the same as $\text{F}\left(\frac{dq}{dx}, q\right) = 0$; and proceeding as in form (3), $dx = \frac{dq}{\text{Q}}$ and $x = \text{Q}' + c$.

Also $\frac{d^2y}{dx} = qdx = \frac{qdq}{\text{Q}}$, $\therefore \frac{dy}{dx} = \text{Q}'' + c'$ and $y = \text{Q}''' + c''$; and eliminating q, the result will be the complete primitive.

Ex. $\frac{d^4yd^3y}{dx^7} = 1$.

Let $r = \frac{d^3y}{dx^3} \therefore \frac{dr}{dx} \times r = 1$, or $dx = rdr \therefore x = \frac{r^2}{2} + c$.

Also $dq = rdx = r^2dr \therefore q = \frac{r^3}{3} + c' \therefore$ . . . . . .

$$dp = qdx = rdr\left\{\frac{r^3}{3} + c'\right\} \therefore p = \frac{r^5}{3.5} + \frac{c'r^2}{1.2} + c'' \therefore \cdots$$

$$dy = rdr\left\{\frac{r^5}{3.5} + \frac{c'r^2}{1.2} + c''\right\} \therefore y = \frac{r^7}{3.5.7} + \frac{c'r^4}{1.2.4} + \frac{c''r^2}{1.2} + c''',$$

in which substitute $r = \sqrt{2(x - c)}$, and there will result the complete primitive.

10. *Required to integrate* $\mathrm{F}\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^{n-2}y}{dx^{n-2}}\right) = 0.$

This is integrable in the same manner as form (5); for let the equation be $\mathrm{F}(r, s, t) = 0$; then since $s = \frac{dr}{dx}$, and $t = \frac{d^2r}{dx^2}$, we have $\mathrm{F}\left(r, \frac{dr}{dx}, \frac{d^2r}{dx^2}\right) = 0$, of which the primitive may be found as in form (5). Let the result be $r = \mathrm{x}$; then if $r = \frac{d^n y}{dx^n}$, we have $\frac{d^n y}{dx^n} = \mathrm{x}$, which has been integrated, art. 6.

11. *An equation of the nth order containing only the independent variable is reducible to one of the* $n - 1$*th order which shall contain both variables.*

For let the equation be $\mathrm{F}(x, p, q, \cdots s, t) = 0$, where $t = \frac{d^n y}{dx^n}$, $s = \frac{d^{n-1}y}{dx^{n-1}}, \cdots$. The equation becomes by substitution $\mathrm{F}\left(x, p, \frac{dp}{dx} \cdots \frac{d^{n-1}p}{dx^{n-1}}\right) = 0$, an equation between x and p of the $n - 1$th order.

Ex. $(1 - x)d^3y - 3dxd^2y = 0$, or $r - \frac{3}{1-x}q = 0$.

Since $r = \frac{dq}{dx}$, we have $\frac{dq}{dx} = \frac{3q}{1-x}$, or $\frac{dq}{q} = \frac{3dx}{1-x} \therefore$ integrating, $q = \frac{c}{(1-x)^3}$, or $\frac{dp}{dx} = \frac{c}{(1-x)^3} \therefore dp = \frac{cdx}{(1-x)^3}$ and $p = \frac{c}{(1-x)^2} + c'$, or $dy = \frac{cdx}{(1-x)^2} + c'dx \therefore$ . . . . .

$y = \frac{c}{1-x} + c'x + c''$, which may take the form . . . . .

$(1 - x)y = cx^2 + c'x + c''$.

12. PRAXIS.

1. $md^2y = (a^2 + bx)dx^2 \therefore my = \frac{a^2x^2}{2} + \frac{bx^3}{2.3} + cx + c'$.

2. $d^2y\sqrt{ay} = dx^2 \therefore x = \frac{2a^{\frac{1}{4}}}{3}(y^{\frac{1}{2}} - 2c^{\frac{1}{2}})\sqrt{y^{\frac{1}{2}} + c^{\frac{1}{2}}} + c'$.

3. $ad^2y = dxdy \therefore y = e^{\frac{x-c}{a}} + c'$.

4. $\frac{d^2y}{dx^2} - \frac{ady^2}{dx^2} + \frac{bdy^3}{dx^3} = 0 \therefore ax - by = ce^{-ay} + c'$.

5. $xd^2y + dxdy = 0 \therefore y = clx + c'$.

6. $xd^2y - dxdy = 0 \therefore c'y = x^2 - c^2$.

7. $(1 - ax^2)d^2y = bxdxdy \therefore y = c\int(1 - ax^2)^{-\frac{b}{2a}}dx$.

8. $2\sqrt{a^2 - ax}dxd^2y + (dx^2 + dy^2)^{\frac{3}{2}} = 0 \therefore$ . . . . .

$y = \sqrt{ax - x^2} - \frac{a}{2}v.s.^{-1}\frac{2x}{a}$.

9. $yd^2y + dy^2 + dx^2 = 0$; a circle, rad. $= \sqrt{a^2 + b^2}$.

10. $xyd^2y = ydxdy + xdy^2 + \frac{bxdy^2}{\sqrt{a^2 - x^2}} \therefore$ . . . . .

$l\frac{(cb + \sqrt{a^2 - x^2})^c}{c'y} = \frac{\sqrt{a^2 - x^2}}{b}$.

11. $x^2d^2y + xdxdy = 4y^3dx^2 \therefore \frac{dx}{x} = -\frac{dy}{\sqrt{2(y^4 - c^4)}}$.

13. The above, together with linear equations, are all the forms which are integable by general processes. Before we proceed to the discussion of linear equations, it will be necessary to consider briefly the method of rendering equations of the higher orders integrable per se by the introduction of a factor. This is Euler's favourite method of integration; but, since we can seldom find this factor without knowing the primitive, it does not possess, in the present state of the science, any great advantage over that of the separation of the variables.

14. It was shown, ch. 4, 21, that when an equation of the first order is not an exact fluxion, it is in consequence

of the disappearance of a factor by the process of differentiation and elimination; and that if this factor can be restored, the equation is integrable. Since an equation of a higher order than the first is derived from one of its first primitives by the same process of differentiation and elimination, it is obvious that the same must be true in this case; and that if the factor can be found, the proposed equation is reducible to one of an inferior order. We shall, however, give in the following article a direct demonstration of this proposition: it is the same as that contained in ch. 4, art. 22, generalized.

15. *An equation of any order whatever between two variables may be always rendered an exact fluxion by the introduction of a factor.*

Let $p, q, \cdots s, t$ be the 1st, 2d, $\cdots n-1$th, nth fluxional coefficients of $y = fx$; then, by solving the equation with respect to t, it may be put under the form $t + f(x, y, p, \cdots r, s) = 0$.

Let c be the arbitrary constant of one of its n first primitives; and let this primitive, when solved with respect to c, be $c = \text{F}(x, y, p, \cdots r, s) =$, by substitution, u, which is therefore a function that does not contain c.

Now in $u = \text{F}(x, y, p, \cdots r, s)$, we may suppose that there are only two variables x and s, which must be made to vary so as to satisfy the equation; and differentiating on this hypothesis, we have $\dot{\frac{du}{dx}}$, which $= 0$, $= \frac{du}{dx} + \frac{du}{ds}\frac{ds}{dx} \cdots = \frac{du}{dx} + \frac{du}{ds}t = u' + u_0 t$, if $u' = \frac{du}{dx}$ and $u_0 = \frac{du}{ds}$, or $\cdots t + \frac{u'}{u_0} = 0$; an equation which is identical with $t + f(x, y, p, \cdots r, s) = 0$, since it contains the same constants; or $t + f(x, y, p, \cdots r, s) = t + \frac{u'}{u_0} \cdots = \frac{1}{u_0}(u' + u_0 t) = \frac{1}{u_0}\dot{\frac{du}{dx}}$; or $\left\{t + f(x, y, p, \cdots r, s)\right\} u_0 = \dot{\frac{du}{dx}}$ which multiplied by dx is an exact fluxion, whose fluent is of the form $u = \varphi(x, y, p \cdots s)$.

It appears then from this investigation that an equation, whatever be its order, when under the form

$t + f(x, y, p \cdots r, s) = 0$ may be always rendered integrable per se by the introduction of a factor; and that this factor $= u_0 = \frac{dc}{ds}$, where $c = \text{F}(x, y, p \cdots s)$ is the arbitrary constant of one of the first primitives.

Cor. 1. Since an equation of the second order contains two first primitives, there are two factors essentially different which render an equation of the second order integrable. Similarly there are n different factors which render integrable an equation of the nth order.

Cor. 2. If z is a factor which renders the equation an exact fluxion and $= du$, then $z\varphi u$, where φ is indeterminate, is also a factor which possesses the same property.

16. *Required to investigate the general form of a factor which will render an equation of the second order integrable per se.*

Let the equation be reduced to the form $q + \text{P} = 0$, where $\text{P} = \text{F}(x, y, p)$; and let t and s be the two factors each of which renders it an exact fluxion, so that $t(q + \text{P}) = \frac{\dot{d}u}{dx}$ and $s(q + \text{P}) = \frac{\dot{d}v}{dx}$; then, $t\varphi' u(q + \text{P}) = \frac{\varphi' u \dot{d}u}{dx} = \frac{\dot{d}.\varphi u}{dx}$ and $s\chi' v(q + \text{P}) = \frac{\dot{d}.\, v}{dx}$.

Let ψ denote the function $\psi\{\varphi u, \chi v\}$; then, we have

$$\left\{ t\frac{d\psi}{du}\varphi' u + s\frac{d\psi}{dv}\chi' v \right\}(q + \text{P}) \cdots$$

$$= \frac{1}{dx}\left\{ \frac{d\psi}{du} d.\varphi u + \frac{d\psi}{dv} d\chi v \right\}; \text{ which } = \frac{\dot{d}\psi}{dx}, \text{ an exact } \cdots$$

fluxion; and consequently the form of the requisite factor is $t\frac{d\psi}{du}\varphi' u + s\frac{d\psi}{dv}\chi' v$.

Ex. 1. $a^2 d^2 y - y dx^2 = 0$, or $q - \frac{y}{a^2} = 0$ (art. 2, ex. 2).

The original primitive is $y = ce^{\frac{x}{a}} + c'e^{-\frac{x}{a}}$; and the two first primitives are $c = \frac{e^{-\frac{x}{a}}}{2}(y + ap)$ and $c' = \frac{e^{\frac{x}{a}}}{2}(y - ap)$; $\therefore$

$\frac{dc}{dp} = \frac{ae^{-\frac{x}{a}}}{2}$ and $\frac{dc'}{dp} = -\frac{ae^{\frac{x}{a}}}{2}$; which are therefore the requisite factors.

Multiplying the proposed by $e^{-\frac{x}{a}}dx$, it becomes $e^{-\frac{x}{a}}\left(dp - \frac{ydx}{a^2}\right)$, which must be an exact fluxion; and it appears from differentiation that it is the fluxion of $\frac{e^{-\frac{x}{a}}}{a}(y + ap)$.

If the proposed be multiplied by $-e^{\frac{x}{a}}dx$, it becomes $e^{\frac{x}{a}}\left(dp - \frac{ydx}{a^2}\right)$, which is the exact fluxion of $\frac{e^{\frac{x}{a}}}{a}(y - ap)$.

Any factor included in either of the forms $e^{-\frac{x}{a}}\phi\left\{e^{-\frac{x}{a}}(y + ap)\right\}$, $e^{\frac{x}{a}}\chi\left\{e^{\frac{x}{a}}(y - ap)\right\}$ will render the proposed an exact fluxion.

Ex. 2. $xd^2y + dxdy = 0$, or $q + \frac{p}{x} = 0$. (Art. 12, ex. 5.)

The complete primitive is $y = clx + c'$; and the two first primitives are $c = px$ and $c' = y - pxlx$; $\therefore$ . . . . .

$\frac{dc}{dp} = x$ and $\frac{dc'}{dp} = -xlx$, which not being included in $x\varphi.px$ is the second independent factor.

Multiplying the proposed by these factors, it becomes $xdp + pdx$, or $d.px = 0$ and $-xlxdp - plxdx$, or $d(y - pxlx) = 0$.

Dividing $c' = y - pxlx$ by $px = c$, and substituting $c'' = \frac{c'}{c}$, we have $c'' = \frac{y}{px} - lx$ $\therefore \frac{dc''}{dp} = -\frac{y}{p^2x}$; which is a third independent factor, since it is not included in either of the forms $x\varphi.px$, and $xlx\chi\{y - pxlx\}$; it must therefore be included in the more general form

$$t\frac{d\psi}{du}\varphi'u + s\frac{d\psi}{dv}\chi'v.$$

To verify this; we have $t = x$, $s = -xlx$, $u = px$, $v = y - pxlx$.

Suppose $\varphi u = u$, $\chi v = v$ $\therefore$ $\varphi'u = 1$ and $\chi'v = 1$; also,

suppose $\psi = \frac{v}{u}$; then $\frac{d\psi}{du} = -\frac{v}{u^2}$ and $\frac{d\psi}{dv} = \frac{1}{u}$; and by substitution, the form becomes $\frac{-xv}{u^2} - \frac{xlx}{u}$, or

$-\frac{xv + xlxu}{u^2}$; which $= -\frac{xy}{p^2x^2} = -\frac{y}{p^2x}$.

17. *Required to investigate the conditions that an equation of the second order may be integrable per se.*

The canonical form of the equation, dx being constant, is $\mathrm{Q}d^2y + \mathrm{R}dxdy + \mathrm{S}dy^2 + \mathrm{T}dx^2 = 0$; of which either of the first primitives is of the form $\mathrm{M}dx + \mathrm{N}dy = \mathrm{C}dx$, where C is the arbitrary constant.

Differentiating this primitive, we have

$\mathrm{N}d^2y + \left\{\frac{d\mathrm{N}}{dx} + \frac{d\mathrm{M}}{dy}\right\}dxdy + \frac{d\mathrm{N}}{dy}dy^2 + \frac{d\mathrm{M}}{dx}dx^2 = 0$; which is identical with the proposed, and consequently $\mathrm{N} = \mathrm{Q}$; $\frac{d\mathrm{N}}{dx} + \frac{d\mathrm{M}}{dy} = \mathrm{R}$; $\frac{d\mathrm{N}}{dy} = \mathrm{S}$; and $\frac{d\mathrm{M}}{dx} = \mathrm{T}$.

From the 1st and 3d we have $\mathrm{S} = \frac{d\mathrm{Q}}{dy}(\alpha)$. Also from the 2d and 4th, $\frac{d\mathrm{R}}{dx} = \frac{d^2\mathrm{N}}{dx^2} + \frac{d^2\mathrm{M}}{dxdy} = \frac{d^2\mathrm{Q}}{dx^2} + \frac{d\mathrm{T}}{dy}(\beta)$; which are therefore the required equations of condition that the proposed equation may be an exact fluxion.

By a similar process equations of condition may be obtained that equations of the third and of the higher orders may be exact fluxions.

18. *Required to integrate an equation of the second order, which is an exact fluxion.*

From the preceding article $\mathrm{N} = \mathrm{Q}$; and to find M, we have $\frac{d\mathrm{M}}{dx} = \mathrm{T}$ and $\frac{d\mathrm{M}}{dy} = \mathrm{R} - \frac{d\mathrm{N}}{dx} = \mathrm{R} - \frac{d\mathrm{Q}}{dx}$; wherefore $d\mathrm{M}$, which $= \frac{d\mathrm{M}}{dx}dx + \frac{d\mathrm{M}}{dy}dy$, $= \mathrm{T}dx + \left\{\mathrm{R} - \frac{d\mathrm{Q}}{dx}\right\}dy$, which is an exact fluxion, because from (β) $\frac{d\mathrm{T}}{dy} = \frac{d\mathrm{R}}{dx} - \frac{d^2\mathrm{Q}}{dx^2}$; and in-

tegrating, $\text{M} = \int \left\{ \text{T}dx + \left(\text{R} - \frac{d\text{Q}}{dx} \right) dy \right\}$.

Cor. If dx be not constant, the canonical form of the equation is $\text{Q}d^2y + \text{R}dxdy + \text{S}dy^2 + \text{T}dx^2 + \text{V}d^2x = 0$; and in order that this may be an exact fluxion, the same conditions must obtain as in the article, and also that

$$\text{V} = \text{M} = \int \left\{ \text{T}dx + \left(\text{R} - \frac{d\text{Q}}{dx} \right) dy \right\}.$$

Ex. 1. $x^2d^2y - xdxdy - 3ydx^2 = 0$.

Here the equations (α) and (β) obtain; and to integrate the proposed, we have $\text{N} = x^2$; also $d\text{M} = -3ydx - 3xdy$, or $\text{M} = c - 3xy$; wherefore the first primitive is of the form $(c - 3xy)dx + x^2dy = 0$, or $dy - \frac{3}{x}ydx = \frac{cdx}{x^2}$, and integrating, $y = x^3 \left\{ \int \frac{cdx}{x^5} + c' \right\} = c'x^3 - \frac{c}{x}$ is the required primitive as in art. 4, ex. 1.

Ex. 2. $xyd^2y + (4xy + y)dxdy + xdy^2 + 2y^2dx^2 = 0$.

Here $\text{N} = xy$ and $d\text{M} = 2y^2dx + 4xydy$, which being homogeneous and an exact fluxion,

$\text{M} = \frac{2y^2x + 4xy^2}{3}$ (ch. 4, 25, ex. 2) $= 2y^2x$; and the first primitive is $2y^2xdx + xydy = cdx$, or $ydy + 2y^2dx = \frac{cdx}{x}$; and by substitution, $dv + 4vdx = \frac{cdx}{x}$; and integrating,

v or $y^2 = e^{-4x} \left\{ \int \frac{ce^{4x}dx}{x} + c' \right\} = c'e^{-4x} + ce^{-4x} \int \frac{e^{4x}dx}{x}$,

which can be integrated only by developement (ch. 1, 39).

Ex. 3. $y^2x^3d^2y + (2y + 3x^2y^2)dxdy + 2x^3ydy^2 + y^2d^2x = 0 = d^2u$.

The equations (α) and (β) obtain; also $d\text{M} = 2ydy$, or $\text{M} = y^2$; wherefore this example is an exact fluxion, and it is the fluxion of $y^2dx + y^2x^3dy = 0 = du$, where $z = \frac{1}{x^3y^2}$.

LINEAR EQUATIONS.

19. *Def.* A linear equation is one in which the function or dependant variable as well as all its fluxional coefficients

are of the first degree. The canonical form is

$\frac{d^n y}{dx^n} + \text{P}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \text{R}\frac{dy}{dx} + \text{S}y = \text{X}$, where P, Q $\cdots$ R, S and X are functions of x.

Linear equations frequently occur in almost every department of physical science. It has been already shown that the primitives of equations of the first order may be exhibited by means of a very simple formula. Equations of the second order are always reducible to those of the first, which however are not linear; and they possess some very peculiar properties which facilitate their integration; but yet there are examples of which the primitives cannot be found by any known methods.

20. *Required to integrate* $\frac{d^n y}{dx^n} + \text{A}\frac{d^{n-1}y}{dx^{n-1}} + \text{B}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + \text{D}\frac{dy}{dx} + \text{E}y = 0$, *where* A, B, $\cdots$ D, E *are constant quantities.*

Substitute $y = e^{ax}$; then, since

$$\left.\begin{aligned}\frac{dy}{dx} &= ae^{ax} = ay\\ \frac{d^2y}{dx^2} &= a^2e^{ax} = a^2y\end{aligned}\right\}$$ the equation becomes, dividing by y, $a^n + \text{A}a^{n-1} + \text{B}a^{n-2} + \cdots + \text{D}a + \text{E} \cdots = 0$ (α).

&c. = &c.

If this can be solved by the Theory of Equations, we can obtain n values of a, and consequently of y. Let them be $y = e^{a'x}, y = e^{a''x}, \cdots y = e^{a^{(n)}x}$; each of which is a solution.

Represent these conjugate values by $y', y'', \cdots y^{(n)}$; then shall the required primitive be of the form

$y = c'y' + c''y'' + \cdots + c^{(n)}y^{(n)}$.

For assume $y = c'y' + c''y'' + \cdots + c^{(n)}y^{(n)}$; then differentiating this assumed value of y, and substituting it in the proposed form, the form becomes

$$\left.\begin{aligned}&c'\left\{\frac{d^n y'}{dx^n} + \text{A}\frac{d^{n-1}y'}{dx^{n-1}} + \cdots + \text{D}\frac{dy'}{dx} + \text{E}y'\right\}\\ &+ c''\left\{\frac{d^n y''}{dx^n} + \text{A}\frac{d^{n-1}y''}{dx^{n-1}} + \cdots + \text{D}\frac{dy''}{dx} + \text{E}y''\right\}\\ &+ \cdots\\ &+ c^{(n)}\left\{\frac{d^n y^{(n)}}{dx^n} + \text{A}\frac{d^{n-1}y^{(n)}}{dx^{n-1}} + \ldots + \text{D}\frac{dy^{(n)}}{dx} + \text{E}y^{(n)}\right\}\end{aligned}\right\}$$

which $= 0$ independently of the values of c', $c'' \cdots c^n$, because each of the forms inclosed in the brackets is (ex hyp.) $= 0$; and consequently the assumed value of y is its true value; and it is the complete primitive, since it contains n arbitrary constants.

21. If the equation (α) contain impossible roots, they lie under the form $a' = \alpha + \beta\sqrt{-1}$ and $a'' = \alpha - \beta\sqrt{-1}$ (Alg. 355), or $y' = e^{(\alpha+\beta\sqrt{-1})x}$ and $y'' = e^{(\alpha-\beta\sqrt{-1})x}$; and the corresponding part of the value of y, which denote by (y), $= ce^{(\alpha+\beta\sqrt{-1})x} + c'e^{(\alpha-\beta\sqrt{-1})x} \cdots$

$= e^{\alpha x}\left\{ce^{\beta x\sqrt{-1}} + c'e^{-\beta x\sqrt{-1}}\right\}$.

But $e^{\beta x\sqrt{-1}} = \cos.\beta x + \sqrt{-1}\,\sin.\beta x$ and $e^{-\beta x\sqrt{-1}} \cdots = \cos.\beta x - \sqrt{-1}\,\sin.\beta x$ (vol. i. ch. 2, 36); wherefore $(y) = e^{\alpha x}\left\{(c + c')\cos.\beta x + (c - c')\sqrt{-1}\,\sin.\beta x\right\}$.

To obtain this under a more convenient form; assume $c + c' = \mathrm{csin.c'}$ and $(c - c')\sqrt{-1} = \mathrm{ccos.c'}$; then we have $(y) = e^{\alpha x}\left\{\mathrm{csin.c'cos.}\beta x + \mathrm{ccos.c'sin.}\beta x\right\} \cdots$

$= \mathrm{c}e^{\alpha x}\sin.(\beta x + \mathrm{c'})$.

Similarly, if there are other impossible roots, that part of y which corresponds to them may be found.

22. If two of the roots of (α) be equal or $a' = a''$, $(y) = (c + c')y' =$, by substitution, cy'; which gives only one arbitrary constant; or the solution fails to be the complete primitive.

To obtain the primitive in this case; first suppose a' and a'' unequal, and that $a'' = a' + h$; then $(y) = c'e^{a'x} + c''e^{(a'+h)x}$

$= e^{a'x}\left\{c' + c''e^{hx}\right\} = e^{a'x}\left\{c' + c''\left(1 + hx + \frac{h^2x^2}{1.2} + \cdots\right)\right\}$

Substitute $c' + c'' = e'$ and $c''h = e''$; or $c'' = \frac{e''}{h}$ and

$c' = e' - \frac{e''}{h}$; wherefore $(y) = e^{a'x}\left\{e' + e''x + \frac{c''h^2x^2}{1.2} + \cdots\right\}$;

and since this satisfies the proposed equation whatever be the values of e', e'', and that h may be assumed of any magnitude whatever, we may suppose $h = 0$ or $a' = a''$; in which case this value of y becomes $e'e^{a'x} + e''xe^{a'x}$; and the

complete primitive is of the form
$y = e'e^{a'x} + e''xe^{a'x} + c'''e^{a'''x} + \cdots + c^n e^{a^n x}$ *.

If *three* roots are equal, or $a' = a'' = a'''$; then, supposing as before $a'' = a' + h$ and $a''' = a' + k$, we have

$(y) = e^{a'x}\{c' + c''e^{hx} + c'''e^{kx}\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$

$= e^{a'x}\Big\{c' + c'' + c''' + c''(hx + \frac{h^2x^2}{1.2} + \text{\&c.}) \cdot \cdot \cdot \cdot$

$+ c'''(kx + \frac{k^2x^2}{1.2} + \text{\&c.})\Big\} =$, by substitution,

$e^{a'x}\{f' + f''x + f'''x^2\}$, when the roots are equal.

The same process may be extended to any number of equal roots.

If the equal roots are impossible, there must be at least four of the form $a' = \alpha + \beta\sqrt{-1} = a''$ and $a''' = \alpha - \beta\sqrt{-1} = a^{\text{iv}}$; and consequently

$(y) = e^{\alpha x}\{(e' + e''x)e^{\beta x\sqrt{-1}} + (e''' + e^{\text{iv}}x)e^{-\beta x\sqrt{-1}}\}$; and it may be shown as in the preceding article, that

$(y) = e^{\alpha x}\{(f' + f''x)\cos.\beta x + (f''' + f^{\text{iv}}x)\sin.\beta x\}$.

Cor. 1. The conjugate values of y are particular solutions; viz. those which result from supposing that one of the constants of the primitive $= 1$, and all the rest $= 0$.

Cor. 2. When two or more roots of the equation (α) are equal, the coefficients of (y), as developed in art. 20, become infinite.

Thus, when two roots are equal, $c'' = \frac{e''}{h}$ and

$c' = e' - \frac{e''}{h}$, both which $= \infty$ when $h = 0$ or $a' = a''$.

The proposition contained in this article may be also demonstrated in the following manner.

23. *Required to investigate the form of the primitive when the equation* (α) *contains equal roots.*

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* As this demonstration, which is D'Alembert's, contains a fallacia suppositionis, the student may assure himself of the truth of the result by ascertaining that $y = e'e^{a'x} + e''xe^{a'x}$ satisfies $\frac{d^2y}{dx^2} + \text{A}\frac{dy}{dx} + \text{B}y = 0$ when $\text{A} = -2a'$ and $\text{B} = a'^2$.

In the proposed form substitute $y = \mathrm{x}e^{ax}$, where x is a function of x whose form is to be determined; then it becomes

$$\left.\begin{aligned} &\mathrm{x}\{a^n + \mathrm{A}a^{n-1} + \mathrm{B}a^{n-2} + \cdots + \mathrm{D}a + \mathrm{E}\} \\ &+ \frac{d\mathrm{x}}{dx}\{na^{n-1} + (n-1)\mathrm{A}a^{n-2} + \cdots + \mathrm{D}\} \\ &+ \frac{d^2\mathrm{x}}{1.2dx^2}\{n(n-1)a^{n-2} + (n-1)(n-2)\mathrm{A}a^{n-3} + \cdots + \mathrm{C}\} \end{aligned}\right\}$$

+ &c.

where each succeeding term inclosed in brackets, considered as an equation, is the limiting equation of the preceding (Alg. 307).

If equation (α) contain no equal root, this form corresponding to any of the values of a cannot $= 0$ unless $\mathrm{x} = a$ constant $= c'$; in which case $\frac{d\mathrm{x}}{dx} = 0$, $\frac{d^2\mathrm{x}}{dx^2} = 0, \cdots$ But if two roots a', a'' be equal, that part of the second term which is inclosed in brackets, considered as an equation has one root $= a'$ (Alg. 319); and consequently in order that the whole form corresponding to the value $a = a'$ may $= 0$, we must have $\frac{d^2\mathrm{x}}{dx^2} = c''$, or $\mathrm{x} = c' + c''x$.

Similarly it may be shown that if (α) contain p equal roots, the form of x must be such that $\frac{d^p\mathrm{x}}{dx^p} = c'^p$, or $\mathrm{x} = c' + c''x + \cdots + c'^p x^{p-1}$; and the required primitive is $y = e^{a'x}\{c' + c''x + \cdots + c'^p x^{p-1}\} + c'^{p+1}e^{a'^{p+1}x} + \cdots$

24. EXAMPLES.

Ex. 1. $\frac{d^2y}{dx^2} + \frac{ady}{dx} + by = 0.$

Substitute $y = e^{mx}$; then the equation (α) is

$m^2 + am + b = 0$; or $m = \frac{-a \pm \sqrt{a^2 - 4b^2}}{2} = -\frac{a}{2} \pm n$ if $n = \frac{\sqrt{a^2 - 4b^2}}{2}$.

Let n^2 be positive or $a > 2b$; then $y' = e^{-\left(\frac{a}{2}+n\right)x}$ and $y'' = e^{-\left(\frac{a}{2}-n\right)x}$, and the required primitive is

$y = ce^{-\left(\frac{a}{2}+n\right)x} + c'e^{-\left(\frac{a}{2}-n\right)x} = e^{-\frac{a}{2}}(ce^{nx} + c'e^{-nx})$.

Let n^2 be negative or the roots of (α) impossible. If $n = \beta\sqrt{-1}$, it may be shown as in the article that

$y = ce^{-\frac{ax}{2}}\sin.(\beta x + c')$.

Let $n^2 = 0$ or the roots of α equal; then

$y = e^{-\frac{ax}{2}}(c + c'x)$.

Ex. 2. $\frac{d^3y}{dx^3} - \frac{3d^2y}{dx^2} + \frac{3dy}{dx} - y = 0$.

Substitute $y = e^{ax} \therefore a^3 - 3a^2 + 3a - 1 = 0$, or $(a-1)^3 = 0$; the roots of which being equal, the primitive is $y = e^x\{c + c'x + c''x^2\}$.

Ex. 3. $\frac{d^4y}{dx^4} - y = 0$.

Substitute $y = e^{ax} \therefore a^4 - 1 = 0$; and the conjugate values of y are $y = e^x$, $y = e^{-x}$, $y = e^{x\sqrt{-1}}$ and $y = e^{-x\sqrt{-1}}$; and the primitive is $y = ce^x + c'e^{-x} \cdots + ce^{x\sqrt{-1}} + c'e^{-x\sqrt{-1}} = ce^x + c'e^{-x} + (c + c')\cos.x \cdots + (c - c')\sqrt{-1}\sin.x = ce^x + c'e^{-x} + c''\sin.(x + c''')$.

25. *The linear equation* $\frac{d^ny}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} \cdots + D\frac{dy}{dx} + Ey = X$ *is always integrable.*

To integrate a particular case; let the proposed be

$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = X$.

Assume $y = e^{mx}\int y'dx$; then

$p = e^{mx}(m\int y'dx + y')$

$q = e^{mx}(m^2\int y'dx + 2my' + p')$, if $p' = \frac{dy'}{dx}$

$r = e^{mx}(m^3\int y'dx + 3m^2y' + 3mp' + q')$, if $q' = \frac{dp'}{dx}$

; and

by substitution, we have

$q' + (3m + A)p' + (3m^2 + 2Am + B)y' + (m^3 + Am^2 + Bm + C)\int y'dx = e^{-mx}X$.

Suppose $m^3 + Am^2 + Bm + C = 0$ (1); then we have $q' + A'p' + B'y' = e^{-mx}X$, where $A' = 3m + A$, $B' = 3m^2 + 2Am + B$.

Again, assume $y' = e^{m'x}\int y''dx$; then

$$\left.\begin{aligned} p' &= e^{m'x}(m'\int y''dx + y'') \\ q' &= e^{m'x}(m'^2\int y''dx + 2m'y'' + p'') \end{aligned}\right\}$$ and our equation becomes by substitution $p'' + (2m' + A')y'' + (m'^2 + A'm' + B')\int y''dx = e^{-(m+m')x}X$.

Suppose $m'^2 + A'm' + B' = 0$ (2); then we have $p'' + A''y'' = e^{-(m+m')x}X$, where $A'' = 2m' + A'$.

Lastly, assume $y'' = e^{m''x}\int y'''dx$; and the equation becomes $y''' + (m'' + A'')\int y'''dx = e^{-(m+m'+m'')x}X$.

Suppose $m'' + A'' = 0$ (3); then

$y''' = e^{-(m+m'+m'')x}X$; wherefore

$y'' = e^{m''x}\int e^{-(m+m'+m'')x}Xdx$; and

$y' = e^{m'x}\int e^{m''x}dx\int e^{-(m+m'+m'')x}Xdx$; and

$y = e^{mx}\int e^{m'x}dx\int e^{m''x}dx\int e^{-(m+m'+m'')x}Xdx$.

Now to determine m, m', m''; we have the equations (1), (2), (3) which are the same as

$$\left.\begin{aligned} &m^3 + Am^2 + Bm + C = 0 \quad (1) \\ &m'^2 + (3m + A)m' + 3m^2 + 2Am + B = 0 \quad (2) \\ &m'' + 2m' + 3m + A = 0 \quad (3). \end{aligned}\right\}$$

To find the values of m, m', m'' which will satisfy these equations; let a, a', a'' be the roots of $z^3 + Az^2 + Bz + C = 0$ in the order of their magnitude.

Assume $m = a$ the least root; diminish the roots by a by substituting $y = z - a$, and the resulting equation is $y^3 + (3a + A)y^2 + (3a^2 + 2Aa + B)y + a^3 + Aa^2 + Ba + C = 0$ (Alg. 282); or, since the last term $= 0$ (ex hyp.), $y^3 + (3a + A)y^2 + (3a^2 + 2Aa + B)y = 0$, where the whole is divisible by y, because one value of $y = 0$. Dividing by y, the resulting equation is $y^2 + (3a + A)y + 3a^2 + 2Aa + B = 0$, which is identical with (2), since $m = a$.

The two values of y are $a' - a$ and $a'' - a$. Assume as before $m' = a' - a$ the least root, and diminishing the roots by $a' - a$ by substituting $x = y - (a' - a)$, the equation becomes $x^2 + \{2(a' - a) + 3a + A\}x + \cdots\cdots (a' - a)^2 + (3a + A)(a' - a) + 3a^2 + 2Aa + B = 0$, or $x + 2(a' - a) + 3a + A = 0$, which is identical with (3), and consequently $m'' = x = a'' - a - (a' - a) = a'' - a'$; or a, $a' - a$ and $a'' - a'$ are values of m, m' and m'' which satisfy (1), (2) and (3), where a, a', a'' are the roots of an equation of the form $a^3 + Aa^2 + Ba + C = 0$.

Substituting these values in the expression for y, we obtain the complete primitive,
$y = e^{ax}\int e^{(a'-a)x}dx\int e^{(a''-a')x}dx\int e^{-a''x}\mathrm{x}dx$, involving three arbitrary constants.

The complete primitive of the proposed equation is
$y = e^{a'x}\int e^{(a''-a')x}dx\int e^{(a'''-a'')x}dx \cdots \int e^{(a'n-a^{n-1})x}dx\int e^{-a'^n}\mathrm{x}dx$;
where a', a'', $\cdots a'^n$ are the roots of
$a^n + \mathrm{A}a^{n-1} + \mathrm{B}a^{n-2} \cdots + \mathrm{D}a + \mathrm{E} = 0$.

If the equation contain equal roots, this solution does not fail. If any of the roots are imaginary, they may be rendered real as in art. 21.

Cor. 1. If $\mathrm{x} = 0$, we have $\int e^{-a''x}\mathrm{x}dx = \int 0 \times dx = c$, because $dc = 0 \times dx$; wherefore $y = e^{ax}\int e^{(a'-a)x}dx\int ce^{(a''-a')x}dx \cdots$
$= e^{ax}\int e^{(a'-a)x}dx\{ce^{(a''-a')x} + c'\} = e^{ax}\{\int ce^{(a''-a)x}dx + \int c'e^{(a'-a)x}dx\}$
$= e^{ax}\{ce^{(a''-a)x} + c'e^{(a'-a)x} + c''\} = ce^{ax} + c'e^{a'x} + c''e^{a''x}$ as in art. 20.

Cor 2. The value obtained for y when separated into single fluents is

$$y = \frac{e^{a'x}\int e^{-a'x}\mathrm{x}dx}{(a'-a'')(a'-a''')\ldots(a'-a'^n)} + \frac{e^{a''x}\int e^{-a''x}\mathrm{x}dx}{(a''-a')(a''-a''')\ldots(a''-a^n)}\cdots$$
$$+ \frac{e^{a^n x}\int e^{-a^n x}\mathrm{x}dx}{(a'^n-a')(a^n-a'')\ldots(a'^n-a'^{n-1})}.$$

26. EXAMPLES.

Ex. 1. $\dfrac{d^2y}{dx^2} + \mathrm{A}\dfrac{dy}{dx} + \mathrm{B}y = \mathrm{R}$.

Let a and b be the roots of $a^2 + \mathrm{A}a + \mathrm{B} = 0$; then the primitive is
$y = e^{ax}\int e^{(b-a)x}dx\int e^{-bx}\mathrm{R}dx = e^{ax}\int e^{(b-a)x}dx\{\int \mathrm{R}e^{-bx}dx + c\}$
$= e^{ax}\int ce^{(b-a)x}dx + e^{ax}\int e^{(b-a)x}dx\int \mathrm{R}e^{-bx}dx$

$$= ce^{bx} + e^{ax}\left\{\frac{e^{(b-a)x}}{b-a}\int \mathrm{R}e^{-bx}dx - \int\frac{\mathrm{R}e^{-ax}dx}{b-a} + c'\right\}$$

$$= ce^{ax} + c'e^{bx} + \frac{1}{a-b}\left\{e^{ax}\int \mathrm{R}e^{-ax}dx - e^{bx}\int \mathrm{R}e^{-bx}dx\right\}.$$

If $a = b$,
$y = e^{ax}\int dx\int \mathrm{R}e^{-ax}dx = e^{ax}\int dx\{\int \mathrm{R}e^{-ax}dx + c\}$
$= cxe^{ax} + e^{ax}\int dx\int \mathrm{R}e^{-ax}dx$
$= cxe^{ax} + e^{ax}\{x\int \mathrm{R}e^{-ax}dx - \int \mathrm{R}e^{-ax}xdx + c'\}$
$= e^{ax}\{c + c'x - \int \mathrm{R}e^{-ax}xdx + x\int \mathrm{R}e^{-ax}dx\}$.

If a and b contain $\sqrt{-1}$, let $a = \alpha - \beta\sqrt{-1}$, $b = \alpha + \beta\sqrt{-1}$; then it may be shown, as in art. 21, that the first part of the value of y is $ce^{\alpha x}(\sin.\beta x + c')$; which is its value when $\mathrm{R} = 0$.

The remainder

$$= \frac{1}{2\beta\sqrt{-1}}\left\{ e^{(\alpha+\beta\sqrt{-1})x} \int \mathrm{R} e^{-(\alpha+\beta\sqrt{-1})x} dx \quad . \quad . \quad . \quad . \quad . \right.$$

$$\left. - e^{(\alpha-\beta\sqrt{-1})x} \int \mathrm{R} e^{-(\alpha-\beta\sqrt{-1})x} dx \right\} \quad . \quad . \quad . \quad .$$

$$= \frac{e^{\alpha x}}{2\beta\sqrt{-1}}\left\{ e^{\beta x\sqrt{-1}} \int e^{-\beta x\sqrt{-1}} e^{-\alpha x} \mathrm{R} dx \quad . \quad . \quad . \quad . \quad . \quad . \right.$$

$$\left. - e^{-\beta x\sqrt{-1}} \int e^{\beta x\sqrt{-1}} e^{-\alpha x} \mathrm{R} dx \right\}.$$

Substituting $e^{\pm\beta x\sqrt{-1}} = \cos.\beta x \pm \sqrt{-1}\sin.\beta x$, this part becomes

$$\frac{e^{\alpha x}}{2\beta\sqrt{-1}}\left\{(\cos.\beta x + \sqrt{-1}\sin.\beta x) \times \quad . \quad . \quad . \quad . \quad . \quad . \quad . \right.$$

$$\int(\cos.\beta x - \sqrt{-1}\sin.\beta x) e^{-\alpha x} \mathrm{R} dx \quad . \quad . \quad . \quad . \quad . \quad .$$

$$-(\cos.\beta x - \sqrt{-1}\sin\beta x) \times \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\left. \int(\cos.\beta x + \sqrt{-1}\sin.\beta x) e^{-\alpha x} \mathrm{R} dx \right\}$$

$$= \frac{e^{\alpha x}}{2\beta\sqrt{-1}}\left\{ -2\cos\beta x \int \sqrt{-1}\sin.\beta x e^{-\alpha x} \mathrm{R} dx \quad . \quad . \quad . \quad . \right.$$

$$\left. + 2\sqrt{-1}\sin.\beta x \int \cos.\beta x e^{-\alpha x} \mathrm{R} dx \right\}$$

$$= \frac{e^{\alpha x}}{\beta}\left\{ \sin.\beta x \int \cos.\beta x . e^{-\alpha x} \mathrm{R} dx - \cos.\beta x \int \sin.\beta x . e^{-\alpha x} \mathrm{R} dx \right\};$$

and the complete primitive is $y = ce^{\alpha x}(\sin.\beta x + c') \quad . \quad . \quad .$

$$+ \frac{e^{\alpha x}}{\beta}\left\{ \sin.\beta x \int \cos.\beta x e^{-\alpha x} \mathrm{R} dx - \cos.\beta x \int \sin.\beta x e^{-\alpha x} \mathrm{R} dx \right\}.$$

Ex. 2. $\dfrac{d^2y}{dx^2} + n^2 y = \mathrm{R}.$

Here $a^2 + n^2 = 0 \therefore a = -n\sqrt{-1}$ and $b = n\sqrt{-1} \therefore$ if $m = n\sqrt{-1}$, we have

$$y = e^{-mx}\int e^{2mx}dx\int \mathrm{R}e^{-mx}dx = e^{-mx}\int e^{2mx}dx\{\int \mathrm{R}e^{-mx}dx + c\}$$
$$= ce^{mx} + e^{-mx}\int e^{2mx}dx\int \mathrm{R}e^{-mx}dx$$
$$= ce^{mx} + \frac{e^{-mx}}{2m}\left\{e^{2mx}\int \mathrm{R}e^{-mx}dx - \int \mathrm{R}e^{mx}dx + c'\right\}$$
$$= ce^{mx} + c'e^{-mx} + \frac{1}{2m}\left\{e^{mx}\int \mathrm{R}e^{-mx}dx - e^{-mx}\int \mathrm{R}e^{mx}dx\right\}.$$

To give this a real form, we have $\alpha = 0$, $\beta = n$ ∴.

$$y = c + c'\sin.nx + \frac{1}{n}\left\{\sin.nx\int\cos nx\,\mathrm{R}dx \quad . \quad . \quad . \quad . \quad . \quad .\right.$$
$$\left. - \cos.nx\int\sin.nx\,\mathrm{R}dx\right\}.$$

If R is expressed in a series in terms of $\sin.x$ and $\cos.x$, the value of y may be found by means of the series demonstrated, ch. 1, 49.

Ex. 3. $\frac{d^2y}{dx^2} + n^2y = 0.$

This may be integrated, and a real form be given to the value of y as in the preceding example. Professor Airy, in his Mathematical Tracts, gives the following integration, which is effected without the introduction of imaginary quantities.

Multiply by $\cos.nxdx$; then

$\cos.\ nx\,.\,\frac{d^2y}{dx^2} + n^2y\cos.nxdx = 0$; and integrating by parts,

$\cos.nx\cdot\frac{dy}{dx} + ny\sin.nx = c$, or $cdx = \cos.nxdy + ny\sin.nxdx.$

The factor necessary to render this an exact fluxion is $\frac{1}{\cos.^2nx}$ (ch. 4, 28, case 3), and we have

$\frac{cdx}{\cos.^2nx} = \frac{dy}{\cos.nx} - \frac{yd.\cos.nx}{\cos.^2nx}$; and integrating,

$\frac{c}{n}\tan.nx = \frac{y}{\cos.nx} - c'$ or $y = \frac{c}{n}\sin.nx + c'\cos.nx.$

This will take the form either of $y = \mathrm{c}\sin.(nx + \mathrm{c}')$, if $\mathrm{c}\cos.\mathrm{c}' = \frac{c}{n}$ and $\mathrm{c}\sin.\mathrm{c}' = c'$; or of $y = \mathrm{c}\cos.(nx + \mathrm{c}')$, if

$\text{ccos.c}' = c'$ and $\text{csin.c}' = -\frac{c}{n}$.

Similarly, if $\frac{d^2y}{dx^2} + n^2y = k$, it may be shown that

$$y = \frac{c}{n}\text{sin.}nx + c'\text{cos.}nx + \frac{k}{n^2} = \text{ccos.}(nx + \text{c}') + \frac{k}{n^2}.$$

Ex. 4. $\frac{d^2y}{dx^2} + y + k\text{cos.}mx = 0.$

Here $(\alpha) = 0$ and $(\beta) = 1 \therefore y = c\text{cos.}x + c'\text{sin.}x \cdots + k\text{cos.}x\int \text{sin.}x\,\text{cos.}mx dx - k\text{sin.}x\int \text{cos.}x\,\text{cos.}mx dx \cdots$

$$= c\text{cos.}x + c'\text{sin.}x - \frac{k\text{cos.}x}{2}\left\{\frac{\text{cos.}(m+1)x}{m+1} - \frac{\text{cos.}(m-1)x}{m-1}\right\}$$

$$- \frac{k\text{sin.}x}{2}\left\{\frac{\text{sin.}(m-1)x}{m-1} + \frac{\text{sin.}(m+1)x}{m+1}\right\} = c\text{cos.}x + c'\text{sin.}x$$

$$- \frac{k}{2}\frac{\text{cos.}mx}{m+1} + \frac{k}{2}\frac{\text{cos.}mx}{m-1} = c\text{cos.}x + c'\text{sin.}x + \frac{k\text{cos.}mx}{m^2-1}.$$

This example may be also integrated without the introduction of imaginary quantities.

These examples belong to the *Problem of the Three Bodies;* for other instances, vid. Woodhouse's Physical Astronomy or Airy's Mathematical Tracts.

Ex. 5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = \text{R}.$

Here $a = a' = a'' = 1 \therefore y = e^x\int dx\int dx\int \text{R}e^{-x}dx.$

If $\text{R} = 0, y = ce^x\int dx\{x + c'\} \cdots$

$$= ce^x\left\{\frac{x^2}{2} + c'x + c''\right\} = e^x\left\{cx^2 + c'x + c''\right\}.$$

27. We have not the means of exhibiting by any known formula the primitive of the general linear equation of the nth order $\frac{d^ny}{dx^n} + \text{P}\frac{d^{n-1}y}{dx^{n-1}} \cdots + \text{s}y = \text{x}\ (\alpha)$. It may be made to depend upon the integration of $\frac{d^ny}{dx^n} + \text{P}\frac{d^{n-1}y}{dx^{n-1}} \cdots + \text{s}y = 0\ (\beta)$; but we cannot integrate this latter when P, Q, $\cdots$ s are variable quantities. Even

the linear equation of the second order $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$, though always reducible to one of the first order, can be integrated but in few instances.

The equation (β) is always reducible to one in which only one of the variables enters.

For, substitute $y = e^z$; then

$$\left.\begin{aligned} dy &= e^z dz = y dz \\ d^2y &= y(dz^2 + d^2z) \\ d^3y &= y(dz^3 + 3dz d^2z + d^3z) \\ \cdots &= \cdots \\ d^ny &= y(dz^n + \frac{n(n-1)}{2} dz^{n-2} d^2z \ldots + d^nz) \end{aligned}\right\}$$

and the equation becomes, dividing by y,

$\frac{d^nz}{dx^n} + P'\frac{d^{n-1}z}{dx^{n-1}} \cdots + R'\frac{dz}{dx} + S' = 0$, an equation which contains only x and the fluxional coefficients; and which must be integrated as in art. 11.

The equation (β) is also reducible to one of the next inferior order.

For, substitute $y = e^{\int z dx}$; then

$$\frac{dy}{dx} = e^{\int z dx} z = yz.$$

$$\frac{d^2y}{dx^2} = y\left\{z^2 + \frac{dz}{dx}\right\}$$

$$\frac{d^3y}{dx^3} = y\left\{z^3 + 3z\frac{dz}{dx} + \frac{d^2z}{dx^2}\right\}$$

$$\cdots = \cdots$$

and if these be substituted in (β), there will result an equation of the $n-1$th order; which, however, is not linear.

28. *Required to integrate the linear equation* . . . . $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.

Substitute $y = e^{\int z dx}$, or $z = \frac{dy}{ydx}$; then

$$\left.\begin{aligned} \frac{dy}{dx} &= yz \\ \frac{d^2y}{dx^2} &= y\left\{z^2 + \frac{dz}{dx}\right\} \end{aligned}\right\}$$; and the equation becomes, dividing by y,

$z^2 + \frac{dz}{dx} + Pz + Q = 0$, or $dz + (z^2 + Pz + Q)dx = 0\ (\beta)$;

an equation of the first order.

If (β) be integrated, the resulting value of y is the complete primitive, since it contains two arbitrary constants; of which the second is introduced by the equation $y = e^{\int z dx}$.

By this substitution all equations of the second order, whether linear or not, which are homogeneous with respect to y, p and q may be reduced to the first order (art. 5).

Cor. 1. If the conjugate values of z in equation (β) can be found, let them be z' and z''; then, $y' = e^{\int z' dx}$ and $y'' = e^{\int z'' dx}$, and the complete primitive is $y = ce^{\int z' dx} + c'e^{\int z'' dx}$.

Cor. 2. If only one of the conjugate values of z or y be known, the complete primitive can be found.

For, substitute $y = vy'$; then the proposed becomes $v(d^2y' + Pdy'dx + Qy'dx) + y'd^2v + 2dy'dv + Py'dvdx = 0$, or (ex hyp.) $y'd^2v + 2dy'dv + Py'dvdx = 0$.

To integrate this, substitute $dv = tdx$ or $\frac{dv}{dx} = t$; then

$y'dt + 2tdy' + Py'tdx = 0$ or $\frac{dt}{t} + \frac{2dy'}{y'} + Pdx = 0$; whence,

integrating, $ty'^2 = ce^{-\int Pdx}$ and $v = \int tdx = c' + \int \frac{ce^{-\int Pdx}dx}{y'^2}$;

which is to be substituted in $y = vy'$.

Cor. 3. If P and Q are constant and $z^2 + Pz + Q$ a perfect square, let it be $(x - a)^2$; then (β) is the same as

$dz + (z - a)^2 dx = 0$, or $\frac{dz}{(z-a)^2} + dx = 0$; wherefore

$x + c = \frac{1}{z-a}$ and $z = a + (x + c)^{-1}$ and $y = e^{\int z dx} \cdot \cdot \cdot$

$= e^{ax + l\frac{x+c}{c'}} = e^{ax}(cx + c')$; which agrees with art. 22.

29. *The form* $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ *is reducible to the form* $\frac{d^2y}{dx^2} + Ny = 0$.

For substitute $y = zt$; then the proposed form becomes

$$t\frac{d^2z}{dx^2}+\left\{2\frac{dt}{dx}+\text{P}t\right\}\frac{dz}{dx}+\left\{\frac{d^2t}{dx^2}+\text{P}\frac{dt}{dx}+\text{Q}t\right\}z=0.$$

Assume $2\frac{dt}{dx}+\text{P}t=0$, or $\frac{dt}{t}+\frac{1}{2}\text{P}dx=0$, or

$t=e^{-\frac{1}{2}\int \text{P}dx}$; then, we have

$t\frac{d^2z}{dx^2}+\left\{\frac{d^2t}{dx^2}+\text{P}\frac{dt}{dx}+\text{Q}t\right\}z=0$; in which, if we substitute $t=e^{-\frac{1}{2}\int \text{P}dx}$, there will result

$\frac{d^2z}{dx^2}+\left\{\text{Q}-\frac{\text{P}^2}{4}-\frac{\frac{1}{2}d\text{P}}{dx}\right\}z=0$; which is of the form

$\frac{d^2y}{dx^2}+\text{N}y=0.$

30. EXAMPLES.

Ex. 1. $\frac{d^2y}{dx^2}-\frac{a^2-1}{4x^2}y=0.$

Substituting $y=e^{\int z dx}$, the equation becomes $dz+z^2dx=\frac{a^2-1}{4}x^{-2}dx$. This is rendered homogeneous by $z=\frac{1}{v}$; when it becomes $-\frac{dv}{v^2}+\frac{dx}{v^2}=\frac{a^2-1}{4}\frac{dx}{x^2}$, or

$-x^2dv+x^2dx=\frac{a^2-1}{4}v^2dx.$

Substitute $v=wx$; then, dividing by x^2, we have

$-(wdx+xdw)+dx=\frac{a^2-1}{4}w^2dx$, or

$$-\frac{dx}{x}=\frac{4}{a^2-1}\,\frac{dw}{w^2+\frac{4w}{a^2-1}-\frac{4}{a^2-1}}.$$

Let m and $-n$ be the roots of $w^2+\frac{4w}{a^2-1}-\frac{4}{a^2-1}=0$; then $n=\frac{2}{a-1}$ and $m=\frac{2}{a+1}$ and

$-\frac{dx}{x}=\frac{1}{a}\left\{\frac{dw}{w-m}-\frac{dw}{w+n}\right\}\therefore\frac{c}{x^a}=\frac{w-m}{w+n}=\frac{1-mxz}{1+nxz}\therefore$

$z = \dfrac{x^a - c}{mx^{a+1} + cnx}$, and the complete primitive is

$y = cx^{\frac{a+1}{2}} + c'x^{\frac{a-1}{2}}$.

Ex. 2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}dx - ny = 0$.

Substitute $y = e^{\int z dx}$; then $dz + \left\{ z^2 + \dfrac{z}{x} - \dfrac{n}{x^2} \right\} dx = 0$,

or $x^2dz = (n - xz - x^2z^2)dx$.

To integrate this, substitute $xz = v$, or $z = \dfrac{v}{x}$ $\therefore$ . . .

$xdv - vdx = (n - v - v^2)dx$, or $\dfrac{ndx}{x} = \dfrac{dv}{n - v^2}$ $\therefore$ . . .

$x^n = c\dfrac{n^{\frac{1}{2}} + xz}{n^{\frac{1}{2}} - xz}$; from which a value of z may be obtained, which must be substituted in $y = e^{\int z dx}$.

Ex. 3. $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + x^2(y - m) = 0$.

Assume $y - m = e^{\int z dx}$; then $\dfrac{dz}{dx} + z^2 + 2xz + x^2 = 0$,

or $(z + x)^2 = -\dfrac{dz}{dx}$ $\therefore (z + x)^2 - 1 = -\dfrac{dz + dx}{dx}$, or

$dx = \dfrac{dz + dx}{1 - (z + x)^2}$ $\therefore$ integrating, $x = \frac{1}{2}l\dfrac{1 + z + x}{c(1 - (z + x))}$, or

$ce^{2x}(1 - (z + x)) = 1 + z + x$ $\therefore z = \dfrac{ce^{2x}(1 - x) - (1 + x)}{ce^{2x} + 1}$

$= -x + \dfrac{ce^{2x} - 1}{ce^{2x} + 1}$ $\therefore \int z dx = \int \dfrac{ce^{2x} - 1}{ce^{2x} + 1}dx - \dfrac{x^2}{2}$ . . . .

$= l(ce^x + c'e^{-x}) - \dfrac{x^2}{2}$ $\therefore y - m = ce^{x - \frac{x^2}{2}} + c'e^{-\left(x + \frac{x^2}{2}\right)}$.

This last step may be also thus deduced. Let z' and z'' be the conjugate values of z, and we have

$\int z'dx = l.e^x - \dfrac{x^2}{2} = x - \dfrac{x^2}{2}$ and $\int z''dx = -x - \dfrac{x^2}{2}$ . . .

$\therefore$ (28, cor. 1) $y - m = ce^{x - \frac{x^2}{2}} + c'e^{-\left(x + \frac{x^2}{2}\right)}$.

Ex. 4. $\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \frac{by}{x^2} = 0.$

If we substitute $y = e^{\int z dx}$, the equation becomes $\frac{dz}{dx} + z^2 + \frac{az}{x} + \frac{b}{x^2} = 0$, which may be rendered homogeneous by substituting $x = \frac{1}{v}$.

Or thus. Substitute $y = e^{\int \frac{z dx}{x}}$; then $\frac{dy}{y} = \frac{z dx}{x}$, or $\frac{dy}{dx} = \frac{zy}{x} \therefore \frac{d^2y}{dx^2} = \frac{z dy + y dz}{x dx} - \frac{zy}{x^2} = \frac{z^2 y}{x^2} + \frac{y dz}{x dx} - \frac{zy}{x^2}$; and the result is $\frac{z^2}{x^2} + \frac{dz}{x dx} - \frac{z}{x^2} + \frac{az}{x^2} + \frac{b}{x^2} = 0$, or $\frac{dx}{x} = -\frac{dz}{z^2 + (a-1)z + b} = -\frac{dz}{(z+m)(z+m')}$; whence the conjugate values of z are $z' = -m$ and $z'' = -m'$, and the required primitive is $y = cx^{-m} + c'x^{-m'}$, where $m + m' = a - 1$ and $mm' = b$.

Ex. 5. $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4n^2 y = 0.$

Substitute $y = e^{\int z dx}$; then $(1+x^2)\left\{z^2 + \frac{dz}{dx}\right\} + xz = 4n^2$, or $dz + z^2 dx + \frac{zx dx}{1+x^2} = \frac{4n^2 dx}{1+x^2}$.

To separate the variables, assume $z = \frac{v}{\sqrt{1+x^2}}$; then $\frac{dv}{\sqrt{1+x^2}} + \frac{v^2 dx}{1+x^2} = \frac{4n^2 dx}{1+x^2} \therefore \frac{dv}{4n^2 - v^2} = \frac{dx}{\sqrt{1+x^2}} \therefore \ldots$ $\frac{2n+v}{2n-v} = c.(x + \sqrt{1+x^2})^{4n} =$, by substitution, $cw^{4n} \therefore$ $\frac{4n}{2n-v} = cw^{4n} + 1 \therefore 2n - v = \frac{4n}{cw^{4n}+1} \therefore \ldots$ $v = \frac{2n(cw^{4n}-1)}{cw^{4n}+1}$.

But $w = x + \sqrt{1+x^2} \therefore \frac{dw}{w} = \frac{dx}{\sqrt{1+x^2}} \therefore z dx$, which

$= \frac{vdx}{\sqrt{1+x^2}}$, $= \frac{2n(cw^{4n}-1)dw}{w(cw^{4n}+1)} = -\frac{2ndw}{w} + \frac{4ncw^{4n-1}dw}{cw^{4n}+1}$ $\therefore$

$y = e^{\int zdx} = \frac{c'(cw^{4n}+1)}{w^{2n}} = cw^{2n} + c'w^{-2n}$, where

$w = x + \sqrt{1+x^2}$.

Ex. 6. $\frac{d^2y}{dx^2} + \frac{\text{P}dy}{dx} + \text{Q}y = 0$, where $\text{P} = \frac{\text{A}}{a+bx}$ and

$\text{Q} = \frac{\text{B}}{(a+bx)^2}$.

Substitute $y = e^{\int zdx}$; then

$\frac{dz}{dx} + z^2 + \frac{\text{A}}{a+bx}, z + \frac{\text{B}}{a+bx^2} = 0$, or

$dz + \left(z^2 + \frac{\text{A}z}{a+bx} + \frac{\text{B}}{(a+bx)^2}\right)dx = 0$ (α).

It is manifest from the form of the proposed equation that y is a function of $\frac{1}{a+bx}$; assume therefore

$z = \frac{m}{a+bx}$, where m is a constant quantity; then

$dz = -\frac{mbdx}{(a+bx)^2}$; wherefore, by substitution in (α), we have $m^2 + (\text{A} - b)m + \text{B} = 0$. Let the roots of this equation be m and m', and we have two conjugate values of y, $y' = e^{\int \frac{mdx}{a+bx}} = e^{\frac{m}{b}l.(a+bx)} = (a+bx)^{\frac{m}{b}}$, and

$y'' = (a+bx)^{\frac{m'}{b}}$; and consequently $y = c(a+bx)^{\frac{m}{b}} \ . \ . \ .$

$+ c'(a+bx)^{\frac{m'}{b}}$, which agrees with ex. 4.

If $m = m'$, this ceases to be the complete primitive; and to obtain it by D'Alembert's process, assume $m = m' + h$; then $y = (a+bx)^{\frac{m}{b}}\left\{c + c'(a+bx)^{\frac{h}{b}}\right\} \ . \ . \ . \ . \ . \ .$

$= (a+bx)^{\frac{m}{b}}\left\{c + c'(1 + l(a+bx).\frac{h}{b} + l(a+bx)^2.\frac{h^2}{2b^2} + \&c.\right\}$

Substitute $c + c' = e'$ and $\frac{c'h}{b} = e''$; $\therefore \ . \ . \ . \ . \ . \ .$

$y=(a+bx)^{\frac{m}{b}}\left\{e' + e''l(a+bx) + l(a+bx)^2\frac{e''h}{2b}+ \&c.\right\}$;

and if we suppose $h=0$, $y=(a+bx)^{\frac{m}{b}}\left\{e' + e''l(a+bx)\right\}$.

If the roots of the equation are impossible, $\alpha+\beta\sqrt{-1}$ and $\alpha-\beta\sqrt{-1}$; then

$$y=(a+bx)^{\frac{\alpha}{b}}\left\{c(a+bx)^{\frac{\beta\sqrt{-1}}{b}} + c'(a+bx)^{\frac{-\beta\sqrt{-1}}{b}}\right\}.$$

But since $e^{\pm x\sqrt{-1}} = \cos.x\pm\sqrt{-1}\sin.x$, substituting for $x, \frac{\beta}{b}l(a+bx)$, we have

$$(a+bx)^{\frac{\pm\beta\sqrt{-1}}{b}} = \cos.\frac{\beta}{b}l(a+bx)\pm\sqrt{-1}\sin.\frac{\beta}{b}l(a+bx);$$

and $y=(a+bx)^{\frac{\alpha}{b}}\left\{\text{ccos.}\frac{\beta}{b}l(a+bx)+\text{c'sin.}\frac{\beta}{b}l(a+bx)\right\}$

$$=\text{D}(a+bx)^{\frac{\alpha}{b}}\sin.\left(\frac{\beta}{b}l(a+bx)+\text{D}'\right).$$

31. *Required to integrate the linear equation* . . . .

$$\frac{d^2y}{dx^2}+\text{P}\frac{dy}{dx}+\text{Q}y=\text{R}.$$

It may be made to depend upon the integration of $\frac{d^2y}{dx^2}+\text{P}\frac{dy}{dx}+\text{Q}y=0$ by the same substitution as that by which the linear equation of the first order $\frac{dy}{dx}+\text{P}y=\text{Q}$ was reduced to the form $\frac{dy}{dx}+\text{P}=0$. (Ch. 4, art. 15.)

For, assume $y=\text{X}z$; then the proposed becomes

$$\text{X}\left\{\frac{d^2z}{dx^2}+\text{P}\frac{dz}{dx}+\text{Q}z\right\}+\frac{2d\text{X}}{dx}\frac{dz}{dx}+\text{P}z\frac{d\text{X}}{dx}+z\frac{d^2\text{X}}{dx^2}=\text{R};$$ and since z and X are undetermined, they may be so assumed that $\frac{d^2z}{dx^2}+\text{P}\frac{dz}{dx}+\text{Q}z=0$ (α); and consequently

$\frac{2dx}{dx}\frac{dz}{dz} + \text{P}z\frac{dx}{dx} + z\frac{d^2x}{dx^2} = \text{R}\ (\beta)$.

Integrating (α), we shall obtain z in terms of x; and to determine x, substitute $\frac{dx}{dx} = p$; then $\frac{d^2x}{dx^2} = \frac{dp}{dx}$ and (β) becomes $dp + \left\{\text{P} + \frac{2dz}{zdx}\right\}pdx = \frac{\text{R}dx}{z}$; which, since P, z and R are functions of x, is a linear equation of the first order; and consequently we have

$$p = e^{-\int\left\{\text{P}dx + \frac{2dz}{z}\right\}}\left\{c + \int\frac{\text{R}}{z}e^{\int\left\{\text{P}dx + \frac{2dz}{z}\right\}}\right\} \quad . \quad . \quad . \quad . \quad .$$

$$= \frac{e^{-\int\text{P}dx}}{z^2}\left\{c + \int\text{R}ze^{\int\text{P}dx}dx\right\}$$; wherefore x, which $= \int pdx + c'$ and y, which $= xz$ may be calculated.

Cor. If we suppose $\text{R} = 0$, the proposed is $\frac{d^2y}{dx^2} + \text{P}\frac{dy}{dx} + \text{Q}y = 0$; and we have $p = \frac{ce^{-\int\text{P}dx}}{z^2}$ and $y = z\{\int pdx + \text{C}'\}$.

The value of z in this case becomes a conjugate value of y, viz. that which results from supposing $\text{C} = 0$ and $\text{C}' = 1$.

Ex. 1. $\frac{d^2y}{dx^2} + \text{A}\frac{dy}{dx} + \text{B}y = \text{R}$.

Here (α) is $\frac{d^2z}{dx^2} + \text{A}\frac{dz}{dx} + \text{B}z = 0$. Let a and b be the roots of $a^2 + \text{A}a + \text{B} = 0$; then one value of z is $z = e^{ax}$; and consequently $p = \frac{e^{-\text{A}x}}{e^{2ax}}\left\{\int\text{R}e^{(\text{A}+a)x} + c\right\}$ . . . . .

$= e^{(b-a)x}\left\{\int\text{R}e^{-bx}dx + c\right\}$ $\therefore y = e^{ax}\int e^{(b-a)x}dx\left\{\int\text{R}e^{-bx}dx + c\right\}$, as in art. 26, ex. 1.

We take only a conjugate value of z: its complete value is $z = ce^{ax} + c'e^{bx}$. The result would have been the same, if we had taken $z = e^{bx}$. If we had substituted the complete value $z = ce^{ax} + c'e^{bx}$, it follows from the theory of arbitrary constants (ch. 3), that one of the constants must have disappeared in $y = xz$; and the result being independent of that one, we see the reason why we are at liberty to suppose either of them $= 0$.

Ex. 2. $\frac{d^2y}{dx^2} + \text{P}\frac{dy}{dx} + \text{Q}y = \text{R}$, where $\text{P} = \frac{\text{A}}{a+bx}$, and $\text{Q} = \frac{\text{B}}{(a+bx)^2}$.

The complete primitive of $\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx} + \text{Q}z = 0$ is $z = c(a+bx)^{\frac{m}{b}} + c'(a+bx)^{\frac{n}{b}}$, where m and n are the roots of $m^2 + (\text{A}-b)m + \text{B} = 0$. Art. 30, ex. 6.

Substitute $z = (a+bx)^{\frac{m}{b}}$; then

$p = (a+bx)^{-\frac{\text{A}+2m}{b}}\left\{\int(a+bx)^{\frac{\text{A}+m}{b}}\text{R}dx + c'\right\}$, wherefore

$$\text{X} = c - \frac{bc'}{\text{A}+2m-b}(a+bx)^{-\frac{\text{A}+2m-b}{b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+\int(a+bx)^{-\frac{\text{A}+2m}{b}}dx\int(a+bx)^{\frac{\text{A}+m}{b}}\text{R}dx, \text{ where } \text{A}-b = -(m+n)$$

$$= c + c'(a+bx)^{\frac{n-m}{b}} + \frac{b}{n-m}\left\{(a+bx)^{\frac{n-m}{b}}\int(a+bx)^{\frac{\text{A}+m}{b}}\text{R}dx \cdots\right.$$

$$\left.-\int(a+bx)^{-\frac{m}{b}+1}\text{R}dx\right\}; \text{ consequently } y = c(a+bx)^{\frac{m}{b}} \quad . \quad . \quad .$$

$$+c'(a+bx)^{\frac{n}{b}} + \frac{b}{m-n}\left\{(a+bx)^{\frac{m}{b}}\int(a+bx)^{-\frac{m}{b}+1}\text{R}dx \quad . \quad . \quad .\right.$$

$$\left.-(a+bx)^{\frac{n}{b}}\int(a+bx)^{-\frac{n}{b}+1}\text{R}dx\right\}.$$

The same value of y would have resulted, if we had taken $z = (a+bx)^{\frac{n}{b}}$.

If $m = n$, $p = (a+bx)^{-1}\left\{\int(a+bx)^{-\frac{m}{b}+1}\text{R}dx + c'\right\}$, wherefore $\text{X} = c + c'l(a+bx) + \int(a+bx)^{-1}dx\int(a+bx)^{-\frac{m}{b}+1}\text{R}dx$

$$= c + c'l(a+bx) + \frac{1}{b}\left\{l(a+bx)\int(a+bx)^{-\frac{m}{b}+1}\text{R}dx \cdots\right.$$

$$\left.-\int l(a+bx).(a+bx)^{-\frac{m}{b}+1}\text{R}dx\right\}$$

and $y = (a+bx)^{\frac{m}{b}}\left\{c + c'l(a+bx)\right\}$ . . . . . .

$$+\frac{(a+bx)^{\frac{m}{b}}}{b}\left\{ l(a+bx)\int(a+bx)^{-\frac{m}{b}+1}\mathrm{R}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \right.$$

$$\left. -\int l(a+bx).(a+bx)^{-\frac{m}{b}+1}\mathrm{R}dx \right\}.$$

We may also realize the value of y when m and n contain $\sqrt{-1}$.

It appears then that we can always find the complete fluent of a linear equation of the second order, $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R}$, if we know one of the conjugate values of $\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx} + \mathrm{Q}z = 0$ (α).

In the present state of the science we have not the means of exhibiting in a formula z in terms of x from equation (α); and consequently we cannot exhibit the primitive of the general linear equation of the second order. Thus far we have succeeded; that if we know the conjugate values of y, or even one of them, in the equation $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q} = 0$, we can always find the primitive of the proposed equation. This property belongs to linear equations of any order whatever, and we shall make it the subject of the next article.

32. *The complete primitive of the linear equation* . . . $\frac{d^ny}{dx^n} + \mathrm{P}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \mathrm{R}\frac{dy}{dx} + \mathrm{S}y = \mathrm{X}$ *shall be* . . . . $y = \mathrm{X}'y' + \mathrm{X}''y'' + \cdots + \mathrm{X}^{(n)}y^{(n)}$, *where* $y', y'', \cdots y^{(n)}$ *are* n *values of* y, *which satisfy the equation when* $\mathrm{X} = 0$, *and* $\mathrm{X}', \mathrm{X}'' \cdots \mathrm{X}^{(n)}$ *are certain functions of* x, *which may be determined.*

For, taking a particular case, let the proposed equation be $d^3y + \mathrm{P}d^2ydx + \mathrm{Q}dydx^2 + \mathrm{R}ydx^3 = \mathrm{X}dx^3$; and let y', y'', y''' be three conjugate values of y which satisfy $d^3y + \mathrm{P}d^2ydx + \mathrm{Q}dydx^2 + \mathrm{R}ydx^3 = 0$.

Assume that $y = py' + qy'' + ry'''$, where p, q and r are functions of x. Differentiating, we have

$dy = pdy' + qdy'' + rdy''' + y'dp + y''dq + y'''dr$.

Now since there are three undetermined variables p, q and r, we may assume for them *three* equations. (Alg. 145.)

First, let $y'dp + y''dq + y'''dr = 0$ (α); which gives $dy = pdy' + qdy'' + rdy'''$.

Differentiating this, we have
$d^2y = pd^2y' + qd^2y'' + rd^2y''' + dpdy' + dqdy'' + drdy'''$.

Next, let $dpdy' + dqdy'' + drdy''' = 0$ (β); which gives $d^2y = pd^2y' + qd^2y'' + rd^2y'''$.

Differentiating this, we have

$d^3y = pd^3y' + qd^3y'' + rd^3y''' + dpd^2y' + dqd^2y'' + drd^2y.'''$

Lastly, assume $dpd^2y' + dqd^2y'' + drd^2y''' = \mathrm{x}dx^3$ (γ); which gives $d^3y = pd^3y' + qd^3y'' + rd^3y''' + \mathrm{x}dx^3$.

Substituting these values of dy, d^2y and d^3y in the proposed form, it becomes

$$\mathrm{x}dx^3 + p\left\{d^3y' + \mathrm{P}d^2y'dx + \mathrm{Q}dy'dx^2 + \mathrm{R}y'dx^3\right\} \cdot \cdot \cdot \cdot$$
$$+ q\left\{d^3y'' + \mathrm{P}d^2y''dx + \mathrm{Q}dy''dx^2 + \mathrm{R}y''dx^3\right\} \cdot \cdot \cdot$$
$$+ r\left\{d^3y''' + \mathrm{P}d^2y'''dx + \mathrm{Q}dy'''dx^2 + \mathrm{R}y'''dx^3\right\} = \mathrm{x}dx^3,$$

which obtains independently of p, q and r, because the forms which are inclosed in the brackets are (ex hyp.) each $= 0$; and consequently $y = py' + qy'' + ry'''$ is a solution of the proposed equation.

To determine p, q and r, we have three equations of condition (α), (β) and (γ); from these the three quantities dp, dq, dr may be each obtained in terms of x by means of elimination; and thus we shall have
$dp = \mathrm{x}'dx$, $dq = \mathrm{x}''dx$ and $dr = \mathrm{x}'''dx$; and integrating, $p = \int \mathrm{x}'dx + c$, $q = \int \mathrm{x}''dx + c'$ and $r = \int \mathrm{x}'''dx + c''$, and consequently

$$y = y'\left\{\int \mathrm{x}'dx + c\right\} + y''\left\{\int \mathrm{x}''dx + c'\right\} + y'''\left\{\int \mathrm{x}'''dx + c''\right\}.$$

The same process being applicable to an equation of any order whatever, it appears that the integration of

$\frac{d^ny}{dx^n} + \mathrm{P}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + \mathrm{R}\frac{dy}{dx} + \mathrm{s}y = \mathrm{x}$ may be always reduced to that of $\frac{d^ny}{dx^n} + \mathrm{P}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + \mathrm{R}\frac{dy}{dx} + \mathrm{s}y = 0$.

Cor. The complete value of y in the proposed equation $=$ its complete value in the equation
$d^3y + \mathrm{P}d^2ydx + \mathrm{Q}dydx^2 + \mathrm{R}ydx^3 = 0 \cdot \cdot \cdot \cdot \cdot \cdot \cdot$
$+ y'\int \mathrm{x}'dx + y''\int \mathrm{x}''dx + y'''\int \mathrm{x}'''dx$.

33. *If only two values of* y *in* $d^3y + \mathrm{P}d^2ydx + \mathrm{Q}dydx^2 + \mathrm{R}ydx^3 = 0$ *be known, the proposed equation may be reduced to one of the second order.*

Let y', y'' be the known values of y.

Substitute in the proposed equation $y = py' + qy''$; then $dy = pdy' + qdy''$, assuming $y'dp + y''dq = 0$; and $d^2y = pd^2y' + qd^2y'' + \mathrm{X}dx^2$, assuming $dpdy' + dqdy'' = \mathrm{X}dx^2$.

And as we have only two undetermined variables, we may not make a third assumption, so that in substituting for d^3y we must take the full value without any equation of condition, viz.

$$d^3y = pd^3y' + qd^3y'' + 2dpd^2y' + 2dqd^2y'' + d^2pdy' + d^2qdy''.$$

Substituting these values of d^3y, d^2y, dy and y, the proposed equation becomes, omitting the forms which are (ex hyp.) $= 0$,

$$2dpd^2y' + 2dqd^2y'' + d^2pdy' + d^2qdy'' + \mathrm{P}.\mathrm{X}dx^3 = \mathrm{X}dx^3.$$

From this equation we can eliminate dq, and consequently d^2q by means of $y'dp + y''dq = 0$; and there will result an equation of the second order containing d^2p, dp and functions of x, which is reducible to the first order by art. 2, form (4).

If only one value y' is known; substituting in the proposed equation $y = py'$, the equation for determining p is

$$\left.\begin{array}{l} y'd^3p + 3dy'd^2p + 3d^2y'dp \\ \quad + \mathrm{P}dx\{y'd^2p + 2dy'dp\} \\ \qquad\quad + \mathrm{Q}dx^2dp \end{array}\right\} = \mathrm{X}dx^3.$$

This is an equation of the third order containing only p and x, and consequently is reducible to one of the second order.

If we suppose $\mathrm{X} = 0$, the conclusions deduced in this article will give the method of integrating the equation $d^3y + \mathrm{P}d^2ydx + \mathrm{Q}dydx^2 + \mathrm{R}ydx^3 = 0$ when we know only two, or even only one of the conjugate values of y.

Since the same process is applicable to an equation of any order whatever, it follows that we can always reduce the general linear equation to one of the first order and degree, if we know $n - 1$ of the values $y', y'', \cdots y^n$. This proposition is due to Lagrange; and the artifice adopted in its demonstration is technically called the *Variation of the Parameters.*

34. *Required to exhibit the complete primitive of* · · ·

$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R}$ *in terms of the conjugate values of* y *in* $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = 0$.

Assume $y = py' + qy''$, and let $v = \frac{y''}{y'}$; then by the method of the variation of the parameters we have

$dy = pdy' + qdy''$; if $y'dp + y''dq = 0$ (α)

$d^2y = pd^2y' + qd^2y'' + \mathrm{R}dx^2$; if $dy'dp + dy''dq = \mathrm{R}dx^2$ (β).

From (α) and (β) we have $(y'dy'' - y''dy')dp = -y''\mathrm{R}dx^2$ or $dp = -\frac{v\mathrm{R}dx^2}{y'dv}$; and integrating, $p = c - \int\frac{v\mathrm{R}dx^2}{y'dv}$.

Also $(y'dy'' - y''dy')dq = y'\mathrm{R}dx^2$, or $dq = \frac{\mathrm{R}dx^2}{y'dv}$ and $q = c' + \int\frac{\mathrm{R}dx^2}{y'dv}$; wherefore

$$y = cy' + c'y'' - y'\int\frac{v\mathrm{R}dx^2}{y'dv} + y''\int\frac{\mathrm{R}dx^2}{y'dv}$$

$$= cy' + c'y'' + y'\int dv\int\frac{\mathrm{R}dx^2}{y'dv}.$$

35. EXAMPLES.

Ex. 1. $\frac{d^2y}{dx^2} + y = b$.

The complete primitive of $d^2y + ydx^2 = 0$ is $y = c\sin.x + c'\cos.x$ (art. 2, ex. 1); or the conjugate values are $y' = \sin.x$, and $y'' = \cos.x$.

Assume $y = p\sin.x + q\cos.x$ $\therefore$ $dy = pd\sin.x + qd\cos.x + \sin.xdp + \cos.xdq = pd\sin.x + qd\cos.x$, if $\sin.xdp + \cos.xdq = 0$ (α).

Differentiating, $d^2y = pd^2\sin.x + qd^2\cos.x + bdx^2$, if $\cos.xdp - \sin.xdq = bdx$ (β).

It is manifest that these values of d^2y and of y satisfy the example.

Now, eliminating dq and dp by means of (α) and (β), we have $dp = b\cos.xdx$ and $dq = -b\sin.xdx$ $\therefore$ . . . . . $p = b\sin.x + c$ and $q = b\cos.x + c'$ and the required primitive is $y = c\sin.x + c'\cos.x + b$. (Art. 26, ex. 3.)

Otherwise. Substitute $y - b = z$; then $\frac{d^2z}{dx^2} + z = 0$, whose primitive is $z = c\sin.x + c'\cos.x$, or $y = c\sin.x + c'\cos.x + b$.

Ex. 2. $\frac{d^2y}{dx^2} - y = b \therefore y = ce^x - c'e^{-x} - b$.

Ex. 3. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{a}{x^2-1}$.

The primitive of $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$ is $y = cx + c'x^{-1}$. (Art. 30, ex. 4.)

Assume $y = px + qx^{-1}$ and we have $xdp + x^{-1}dq = 0$ (α) and $dp - x^{-2}dq = \frac{adx}{x^2-1}$ (β), $\therefore dp = \frac{\frac{1}{2}adx}{x^2-1}$ and

$dq = -\frac{\frac{1}{2}ax^2dx}{x^2-1} \therefore p = \frac{a}{4}\, l\,\frac{x-1}{x+1} + c$ and

$q = c' - \frac{a}{2}\left\{x + \frac{1}{2}l\frac{x-1}{x+1}\right\}$ (vol. i. ch. 2, 61, praxis 5), $\therefore$

$y = cx + c'x^{-1} - \frac{ax}{2} + \frac{a}{4}(x - x^{-1})\, l\,\frac{x-1}{x+1}$ . . . .

$= cx + c'x^{-1} + \frac{a(x^2-1)}{4x}\, l\,\frac{x-1}{x+1}$.

Ex. 4. $\frac{d^2y}{dx^2} - \frac{a^2-1}{4x^2}y = mx^{-\frac{a+1}{2}}$.

The primitive of $\frac{d^2y}{dx^2} - \frac{a^2-1}{4x^2}y = 0$ is $y = cx^{\frac{a+1}{2}} + c'x^{-\frac{a-1}{2}}$ (art. 30, ex. 1).

Assume $y = px^{\frac{a+1}{2}} + qx^{-\frac{a-1}{2}}$; and we have $x^{\frac{a+1}{2}}dp + x^{-\frac{a-1}{2}}dq = 0$, or $x^adp + dq = 0$ (α); and $\frac{a+1}{2}x^{\frac{a-1}{2}}dp - \frac{a-1}{2}x^{-\frac{a+1}{2}}dq = mx^{-\frac{a+1}{2}}dx$, or

$(a+1)x^adp - (a-1)dq = 2mdx$ (β) $\therefore dp = \frac{mdx}{ax^a}$ and

$dq = -\frac{mdx}{a}$ $\therefore p = c - \frac{m}{a(a-1)x^{a-1}}$ and $q = c' - \frac{mx}{a}$ $\therefore$

$$y = cx^{\frac{a+1}{2}} + c'x^{-\frac{a-1}{2}} - \frac{mx^{-\frac{a-3}{2}}}{a(a-1)} - \frac{mx^{-\frac{a-3}{2}}}{a} \quad . \quad . \quad . \quad . \quad . \quad .$$

$$= cx^{\frac{a+1}{2}} + c'x^{-\frac{a-1}{2}} - \frac{mx^{-\frac{a-3}{2}}}{a-1} = \frac{1}{x^{\frac{a-1}{2}}}\left\{c' + cx^a - \frac{mx}{a-1}\right\}.$$

Ex. 5. $\frac{d^4y}{dx^4} - y = a^x$.

The primitive of $\frac{d^4y}{dx^4} - y = 0$ is

$y = ce^x + c'e^{-x} + c''\sin.x + c'''\cos.x$.

Assume $y = pe^x + qe^{-x} + r\sin.x + s\cos.x$; then we have by successive differentiations and assumptions

$$\left.\begin{array}{l} e^x dp + e^{-x}dq + \sin.x dr + \cos.x ds = 0 \ (\alpha) \\ e^x dp - e^{-x}dq + \cos.x dr - \sin.x ds = 0 \ (\beta) \\ e^x dp + e^{-x}dq - \sin.x dr - \cos.x ds = 0 \ (\gamma) \\ e^x dp - e^{-x}dq - \cos.x dr + \sin.x ds = a^x \ (\delta) \end{array}\right\}. \text{ For}$$

$$\left.\begin{array}{l} \frac{dy}{dx} = pe^x - qe^{-x} + r\cos.x - s\sin.x \\ \frac{d^2y}{dx^2} = pe^x + qe^{-x} - r\sin.x - s\cos.x \\ \frac{d^3y}{dx^3} = pe^x - qe^{-x} - r\cos.x + s\sin.x \end{array}\right\}.$$

Add (α) and (γ) $\therefore e^x dp + e^{-x}dq = 0$
Add (β) and (δ) $\therefore e^x dp - e^{-x}dq = \frac{1}{2}a^x$ $\therefore dp = \frac{1}{4}a^x e^{-x}dx$

$= \frac{1}{4}e^{(A-1)x}dx$, if $A = la$; $\therefore p = c + \frac{1}{4(A-1)}e^{(A-1)x}$.

Similarly, $q = c' - \frac{1}{4(A+1)}e^{(A+1)x}$.

Subtract (γ) from (α) $\therefore \sin.x dr + \cos.x ds = 0$
Subtract (δ) from (β) $\therefore \cos.x dr - \sin.x ds = -\frac{1}{2}a^x$ $\therefore$

$dr = -\frac{1}{2}a^x\cos.x dx$ $\therefore$

$$r = c'' - \frac{a^x}{2(A^2+1)}\{A\cos.x - \sin.x\} \quad \text{(ch. 1, 57)}$$

$ds = \frac{1}{2}a^x\sin.x dx$ $\therefore$

$$s = c''' + \frac{a^x}{2(A^2+1)}\left\{A\sin.x - \cos.x\right\}; \text{ wherefore}$$

$$y = ce^x + c'e^{-x} + c'''\cos.x + \frac{a^x}{A^4-1}$$

$$= ce^x + c'e^{-x} + c''(\sin.x + c''') + \frac{a^x}{A^4-1}.$$

This example may be also integrated by the formula, art. 25.

36. We have hitherto supposed the fluxional equation to have been deduced on the hypothesis that x is the principal variable. In the application of the calculus, formulæ in which the fluxion of neither variable is constant rarely occur, and we shall soon show that such formulæ have, in general, no certain and definite meaning. In the equations which do occur, if the question relates to curves, one of the co-ordinates is taken as the principal variable; and in some cases, a function of these, such as the area of the curve, or its length, or the solid of resolution; in which case it may be called the principal or independent function. Thus if the area of the curve be the independent function, we have $d.ydx = 0$; if the length, $d.\sqrt{dx^2 + dy^2} = 0$.

In all problems that must be taken as the independent function which will enable us to deduce the fluxional equation from the conditions of the problem; and it may happen that the resulting equation cannot be integrated, or at least that the problem cannot be solved, unless the independent variable be previously changed. Hence arises the necessity of changing this variable; and we shall conclude the chapter with establishing a few rules for that purpose.

37. *Given an implicit function of two variables* $= 0$; *required to find the fluxional coefficients of either variable considered as a function of the other.*

Let $F(x, y) = 0$; then, solving this equation, we have both $y = fx$ and $x = Fy$; wherefore, if $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$, $r = \frac{d^3y}{dx^3}$, $\cdots$ $p' = \frac{dx}{dy}$, $q' = \frac{d^2x}{dy^2}$, $r' = \frac{d^3x}{dy^3}$ $\cdots$, we have

$$k = p\frac{h}{1} + q\frac{h^2}{1.2} + r\frac{h^3}{1.2.3}\cdots$$

and $h = p'\frac{k}{1} + q'\frac{k^2}{1.2} + r'\frac{k^3}{1.2.3}\cdots$ } whence, by substitution,

$$k = p\left\{ p'\frac{k}{1} + q'\frac{k^2}{1.2} + r'\frac{k^3}{1.2.3}\cdots \right\}$$
$$+ \frac{q}{1.2}\left\{ p'^2k^2 + p'q'k^3\cdots \right\}$$
$$+ \frac{r}{1.2.3}\left\{ p'^3k^3 + \cdots \right\}$$
$$+ \&c.;$$

and arranging the terms according to the powers of k, and dividing by k,

$$1 = pp' + (pq' + qp'^2)\frac{k}{1.2} + (pr' + 3qp'q' + rp'^3)\frac{k^2}{1.2.3} + \cdots$$

But k is an independent quantity; and consequently (Alg. 347),

$1 = pp'$ or $p' = \frac{1}{p}$ as in vol. i. ch. 4, 14,

$$0 = pq' + qp'^2 \text{ or } q' = -\frac{qp'^2}{p} = -\frac{q}{p^3}$$

$$0 = pr' + 3qp'q' + rp'^3 \text{ or } r' = \frac{3q^2 - pr}{p^5};\ \&c.$$

From the same equations p, q, r... may be found in terms of p', q', r'... and there will result

$$p = \frac{1}{p'},\ q = -\frac{q'}{p'^3},\ r = \frac{3q'^2 - p'r'}{p'^5}\cdots$$

Cor. These equations may be also deduced immediately by the process of differentiation.

For, from the notation $\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} =$, if dy and dx both vary, $\frac{dxd^2y - dyd^2x}{dx^3} = \frac{d^2y}{dx^2} - \frac{dy}{dx}\frac{d^2x}{dx^2} =$, if dy be constant, $-\frac{dy}{dx}\frac{d^2x}{dx^2}$.

Hence the rules. (1) 'To transform any formula in which d^2y alone enters into one in which d^2x shall alone enter; substitute for $\frac{d^2y}{dx^2}$ its equivalent $-\frac{dyd^2x}{dx^3}$.'

(2) 'To transform any formula in which d^2y alone enters into one in which both d^2y and d^2x shall enter, substitute for $\frac{d^2y}{dx^2}$ its equivalent $\frac{d^2y}{dx^2}-\frac{dyd^2x}{dx^3}$.'

Similarly it may be shown that if d^3y alone enters the formula, it may be transformed into one in which both d^3y and d^3x shall enter by substituting

$$\frac{d^3y}{dx^3}=\frac{d^3y}{dx^3}-\frac{3d^2yd^2x}{dx^4}+\frac{3dyd^2x^2}{dx^5}-\frac{dyd^3x}{dx^4}.$$

38. *Required to find the fluxional coefficients of $y=fx$ in terms of the fluxional coefficients of y and x, each considered as a function of a third variable t.*

There are two equations and three variables; and consequently by elimination each of the variables may become a function of either of the others. (Vol. i. ch. 4, 35.)

Let $p, q, r, \cdots$ be as before the fluxional coefficients of $y=fx$.

Also, let $y'=\frac{dy}{dt}, y''=\frac{d^2y}{dt^2}, \cdots$ and

$$x'=\frac{dx}{dt}, x''=\frac{d^2x}{dt^2}, \cdots; \text{ then}$$

$$\left.\begin{aligned} k&=p\cdot\frac{h}{1}+q\cdot\frac{h^2}{1.2}+r\cdot\frac{h^3}{1.2\,3}+\cdots\\ \text{and } h&=x'\cdot\frac{g}{1}+x''\cdot\frac{g^2}{1.2}+x'''\cdot\frac{g^3}{1.2.3}+\cdots\end{aligned}\right\}\text{, whence, by substitution,}$$

$$\begin{aligned} k=p&\left\{x'\cdot\frac{g}{1}+x''\cdot\frac{g^2}{1.2}+x'''\cdot\frac{g^3}{1.2.3}\cdots\right\}\\ &+\frac{q}{1.2}\left\{x'^2g^2+x'x''g^3\cdot\ \cdot\ \cdot\right\}\\ &+\frac{r}{1.2.3}\left\{x'^3g^3+\cdots\right\}\\ &+\text{\&c.}\end{aligned}$$

Also $k=y'\cdot\frac{g}{1}+y''\cdot\frac{g^2}{1.2}+y'''\cdot\frac{g^3}{1.2.3}\cdots$; wherefore, equating like terms,

$$px' = y' \text{ or } p = \frac{y'}{x'}.$$

$$px'' + qx'^2 = y'' \text{ or } q = \frac{y'' - x''p}{x'^2} = \frac{x'y'' - y'x''}{x'^3}.$$

$$px''' + 3qx'x'' + rx'^3 = y''' \text{ or}$$

$$r = \frac{x''y''' - 3x'x''y''' + 3y'x''^2 - x'y'x'''}{x'^5} \text{ \&c.}$$

Cor. If t be assumed $= y$, *i. e.* if y be taken as the independent variable, we have $y' = 1$; also $y'' = y''' = \cdots = 0$; and the values of $p, q, r, \cdots$ coincide with those deduced in the preceding article.

The values of $p, q, r, \cdots$ might have been deduced immediately by differentiation without the aid of Taylor's Theorem. By means of these values the principal variable may be changed from x to any proposed function of (x, y). If the two equations between y, x and t are under the form $t = \text{F}(x, y)$ and $t = f(x, y)$, it will not be necessary to eliminate t in order to obtain the values of $y', y'', \ldots$ and of $x', x'' \ldots$

Before we proceed to the consideration of formulæ which have no independent function, we shall give one or two instances of changing the independent function by differentiation solely as in art. 37, cor.

39. *Required to transform a formula of the second order in which* $\sqrt{dx^2 + dy^2} = ds$ *is constant into one in which* dx *shall be constant.*

Let d^2y enter into the proposed formula; then

$$\frac{d^2y}{ds^2} = \frac{d\frac{dy}{ds}}{ds} =, \text{ if } ds \text{ vary, } \frac{d^2y}{ds^2} - \frac{dyd^2s}{ds^3}. \text{ But (ex hyp.)}$$

$$d^2s = \frac{dyd^2y}{\sqrt{dx^2 + dy^2}} = \frac{dyd^2y}{ds}; \text{ wherefore}$$

$$\frac{(d^2y)}{ds^2} = \frac{d^2y}{ds^2} - \frac{dy^2d^2y}{ds^4} = \frac{dx^2d^2y}{ds^4}; \text{ whence the rule. 'Sub-}$$

stitute for d^2y its equivalent $\frac{dx^2}{ds^2}d^2y$.'

If d^2x also enters into the formula, we have

$\frac{d^2x}{ds^2} = \frac{d\frac{dx}{ds}}{ds} =$, if dx be constant and ds vary,

$$-\frac{dxd^2s}{ds^3} = -\frac{dxdyd^2y}{ds^4} = -\frac{pq}{(1+p^2)^2}, \text{ or } \frac{(d^2x)}{dx^2} = -\frac{pq}{1+p^2}$$

which must therefore be substituted for $\frac{d^2x}{dx^2}$.

If d^2x enters into the proposed formula and not d^2y, and it be required to transform it into one in which dy shall be constant, the rule is, 'Substitute for d^2x its equivalent $\frac{dy^2}{ds^2}d^2x$.'

40. *Required to transform a formula of the second order in which dx is constant into one in which ds shall be constant.*

First change the formula into one in which dx shall not be constant: this is effected by substituting

$\frac{d^2y}{dx^2} = \frac{d^2y}{dx^2} - \frac{dyd^2x}{dx^3}$ (37, cor. rule 2).

But, since $ds = \sqrt{dx^2 + dy^2} =$ constant, we have

$0 = dxd^2x + dyd^2y$, or $\frac{d^2y}{dx^2} = -\frac{d^2x}{dxdy}$; wherefore, for $\frac{d^2y}{dx^2}$ in the proposed formula we are to substitute

$-\frac{d^2x}{dxdy} - \frac{dyd^2x}{dx^3}$, which $= -\frac{ds^2d^2x}{dx^3dy}$. Also, since y is to continue to be the dependent variable, we must again substitute $d^2x = -\frac{dyd^2y}{dx}$; and there results $\frac{(d^2y)}{dx^2} = \frac{ds^2d^2y}{dx^4}$, which agrees with the preceding article.

By a similar process the independent function may be changed, when there are only two variables, in any way that may be required.

41. EXAMPLES.

Ex. 1. $dx^2dy - dy^3 = adxd^2y + xdxd^2y = (a+x)dxd^2y = xdxd^2y$, if the origin be changed.

Transform this equation into one in which y shall be the independent variable; then by art. 37, cor. rule 1, we have $dx^2dy - dy^3 = -xdyd^2x$, or $xd^2x + dx^2 = dy^2$; an equation in which dy is constant. Let $dy = c$; then $xd^2x + dx^2 = cdy$, and integrating, $xdx = cy + c' = ydy + c'dy$; and again integrating, $x^2 = y^2 + cy + c'$, and the required primitive is $x^2 + 2ax = y^2 + cy + c'$.

Ex. 2. $xd^2y - dxdy + xdy^2 = 0$. (Art. 2, (4), ex. 1.)

Substituting for d^2y its equivalent $-\frac{dyd^2x}{dx}$, there results $\frac{xdyd^2x}{dx} + dxdy = xdy^2$, or $\frac{d^2x}{dx} + \frac{dx}{x} = dy$; and integrating, $l.\frac{xdx}{cdy} = y$; for in the first integration dy is constant; whence $xdx = ce^ydy$, and again integrating, $x^2 + c'^2 = ce^y$.

In the following examples s is the independent variable.

Ex. 3. $\frac{ds^2d^2y}{dx^4} = \frac{1}{a}\cos.\frac{x}{b}$.

Here $\frac{d^2y}{dx^2} = \frac{1}{a}\cos.\frac{x}{b}\frac{dx^2}{ds^2}$; wherefore, by substitution, $\frac{d^2y}{ds^2} = \frac{1}{a}\cos.\frac{x}{b}\frac{dx^2}{ds^2}$, or $\frac{d^2y}{dx} = \frac{1}{a}\cos.\frac{x}{b}dx$; $\therefore$ integrating, $\frac{dy}{dx} = \frac{b}{a}\sin.\frac{x}{b} + c$, and the required primitive is

$$ay + b^2\cos.\frac{x}{b} = cx + c'^2.$$

Ex. 4. $dsdy^2 = adxd^2x$.

Interchanging the co-ordinates for the sake of convenience, we have $ds = \frac{adyd^2y}{dx^2} =$, by substitution, $\frac{adyd^2y}{ds^2}$, where x is the independent variable; whence

$$(1 + p^2)^{\frac{3}{2}} = apq = \frac{apdp}{dx}, \text{ or } dx = \frac{apdp}{(1+p^2)^{\frac{3}{2}}} \therefore \ldots$$

$$x = -\frac{a}{\sqrt{1+p^2}} + c.$$

Also $dy = pdx = \frac{ap^2dp}{(1+p^2)^{\frac{3}{2}}}$ ∴ . . . . . .

$y = a\left\{l(p + \sqrt{1+p^2}) - \frac{p}{\sqrt{1+p^2}}\right\} + c'$; from which two equations eliminating p there will result the required primitive, remembering to restore the co-ordinates.

Ex. 5. $ads d^2y + ydy^2dx = 0$.

If we were to substitute for d^2y according to the rule, there would result an equation between y, p and q, from which both y and x may be obtained in terms of p; but as the elimination of p is in this case difficult, we shall integrate this example by a method which is independent of the preceding rules. Dividing by $dydx$, we have

$\frac{ads d^2y}{dydx} + ydy = 0$, or $\frac{ads d^2y}{dy\sqrt{ds^2 - dy^2}} + ydy = 0$; ∴ integrating,

$al.\frac{ds - dx}{cdy} + \frac{y^2}{2} = 0$ ∴ $ds - dx = ce^{-\frac{y^2}{2a}}dy$; and reducing this, $dx = \left\{\frac{e^{\frac{y^2}{2a}}}{c} - ce^{-\frac{y^2}{2a}}\right\}\frac{dy}{2}$, from which x may be obtained in a series in terms of y. (Ch. 2, 14, ex. 11.)

Ex. 6. $dx^3dy - xds^2d^2y = adxds\sqrt{d^2x^2 + d^2y^2}$.

Here d^2y and d^2x must both be adapted to the hypothesis that dx is constant; in which case the equation becomes

$p - xq = aq$, or $p - \frac{xdp}{dx} = \frac{adp}{dx}$ or $\frac{dx}{a+x} = \frac{dp}{p}$ ∴ . . .

$p = \frac{a+x}{c}$ and $dy = pdx = \frac{(a+x)dx}{c}$ ∴ $cy = 2ax + x^2 + c'^2$.

Ex. 7. $ydyd^2x - ydxd^2y - dx^3 - dxdy^2 = 0$.

Here $y(dyd^2x - dxd^2y) - dxds^2 = 0$; but since ds is constant, $d^2x = -\frac{dyd^2y}{dx}$ ∴ by substitution

$-y\left(\frac{dy^2d^2y}{dx} + dxd^2y\right) - dxds^2 = 0$, or $-\frac{yd^2y}{dx} = dx$, or $yd^2y - dy^2 + ds^2 = 0$.

Let $p = \frac{dy}{ds}$, $q = \frac{d^2y}{ds^2}$; then $yq - p^2 + 1 = 0$, or

$q = \frac{1-p^2}{y}$; but (art. 2. (5)) $q = \frac{pdp}{dy} \therefore \frac{dy}{y} = \frac{pdp}{1-p^2}$; and integrating, $y = \frac{a}{\sqrt{1-p^2}} = \frac{ads}{dx}$, which belongs to a circle.

42. *An homogeneous formula of a higher order than the first in which the fluxion of neither variable is constant, may be always changed into one which shall contain only* p, q, $r, \ldots$ *and* dx, *whatever be the function which is taken to be the independent function.*

To show this by examples.

First, let $\int y dx$ be the independent function.

Here ydx is constant, or $yd^2x + dydx = 0$, or

$$d^2x = -\frac{dydx}{y} = -\frac{p}{y}dx^2.$$

Differentiating, $d^3x = -\frac{qdx^3}{y} + \frac{2pdx}{y} \cdot \frac{p}{y}dx^2 + \frac{p^2dx^3}{y^2} \cdots$

$$= \left\{ -\frac{q}{y} + \frac{3p^2}{y^2} \right\} dx^3.$$

Similarly $d^4x = \cdots$

$$\left\{ -\frac{r}{y} + \frac{7pq}{y^2} - \frac{6p^3}{y^3} \right\} dx^4 - \left\{ \frac{-q}{y} + \frac{3p^2}{y^2} \right\} \frac{3pdx^4}{y} \cdots$$

$$= \left\{ -\frac{r}{y} + \frac{10pq}{y^2} - \frac{15p^3}{y^3} \right\} dx^4$$

&c. = &c.

If these values of d^2x, d^3x, $\cdots$ be substituted in the proposed formula, and the whole be divided by the highest power of dx, it will be changed into one which shall contain only p, q, r, $\cdots$ and dx.

Next, let $\int \sqrt{dx^2 + dy^2}$ be the independent function.

Here $dx\sqrt{1+p^2}$ is constant; and it may be shown that

$$d^2x = -\frac{pq}{1+p^2}dx^2$$

$$d^3x = \left\{ -\frac{pr}{1+p^2} + \frac{(3p^2-1)q^2}{(1+p^2)^2} \right\} dx^3$$

$$d^4x = \left\{ -\frac{ps}{1+p^2} + \frac{(10p^2-3)qr}{(1+p^2)^2} - \frac{(15p^2-13)p^3q}{(1+p^2)^3} \right\} dx^4$$

&c. = &c.

Whatever then is taken to be the independent function, which is always decided upon at the outset of the calculation, the formula may be changed into one which shall contain only p, q, $r \cdots$ and dx.

43. *A formula of a higher order than the first, which has no independent function, has,* in general, *an uncertain and indeterminate value which is different if different independent functions be taken.*

The truth of this proposition appears from examples.

Ex. 1. Let the formula contain d^2x; then if x be taken to be the independent function, $d^2x = 0$; if x^2, $2xdx$ = a constant, and differentiating $xd^2x + dx^2 = 0$, or

$d^2x = -\dfrac{dx^2}{x}$; if x^n, $x^{n-1}d^2x + (n-1)x^{n-2}dx^2 = 0$, or

$d^2x = -\dfrac{(n-1)dx^2}{x}$, which is different according to the different values of n.

Ex. 2. $\dfrac{yd^2x + xd^2y}{dx^2} = du.$

If x be the independent variable, $du = \dfrac{xd^2y}{dx^2}$; if y, $\cdots$ $du = \dfrac{yd^2x}{dx^2}$.

Now these do not represent equal quantities; for if they did, they would be the same whatever function y is of x. Suppose y any function whatever of x, $y = x$ excepted, and it will appear that the values are different. Thus, let $y = x^2$; then $\dfrac{xd^2y}{dx^2} = 2x$; but $\dfrac{yd^2x}{dx^2} = -x$.

Hence we conclude that the value of the proposed example is uncertain and indeterminate, and that it depends upon the function which is taken to be the independent function.

44. There are, however, some formulæ which possess the same value whichever is taken to be the independent function.

Ex. $\dfrac{dyd^2x - dxd^2y}{dx^3} = du.$

If x be the independent variable, $du = -\dfrac{d^2y}{dx^2}$; if y, $\cdots$

$du = \frac{dyd^2x}{dx^3}$; if $\int ydx$, since $yd^2x + dydx = 0$, ...

$du = -\frac{d^2y}{dx^2} - \frac{dy^2}{ydx^2}$.

Now all these possess the same value whatever function y is of x; for let $y = x^n$, then $-\frac{d^2y}{dx^2} = -n(n-1)x^{n-2}$. Also $dy = nx^{n-1}dx$; and differentiating, y being the independent variable, $0 = nx^{n-1}d^2x + n(n-1)x^{n-2}dx^2$, or

$d^2x = -\frac{(n-1)dx^2}{x}$; wherefore $\frac{dyd^2x}{dx^3} = \frac{dy}{dx}\frac{d^2x}{dx^2}$, . . .

$= -n(n-1)x^{n-2}$. Again, since $dy = nx^{n-1}dx$, we have $ydy = nx^{n-1}.ydx$; and differentiating, $\int ydx$ being the independent function,

$yd^2y + dy^2 = n(n-1)x^{n-2}.ydx^2$, or $-\frac{d^2y}{dx^2} - \frac{dy^2}{ydx^2}$ . . . .

$= -n(n-1)x^{n-2}$.

Suppose $y = e^x$; then $d^2y = e^xd^2x + e^xdx^2$; and we have on the three hypotheses $-e^x = -\frac{d^2y}{dx^2} = \frac{dyd^2x}{dx^3}$ . . . .

$= -\frac{d^2y}{dx^2} - \frac{dy^2}{ydx^2}$.

Suppose $y = \sqrt{1-x^2}$; and there will result as before,

$(1-x^2)^{-\frac{3}{2}} = -\frac{d^2y}{dx^2} = \frac{dyd^2x}{dx^3} = -\frac{d^2y}{dx^2} - \frac{dy^2}{ydx^2}$.

And whatever function y is of x, this formula possesses a certain value which is not affected by the independent variable.

Hence it appears that there are certain formulæ in which the fluxion of neither variable is constant, which possess a determinate value; and that in order to find this value, we are at liberty to suppose either variable or any function of them to be the independent function, according as it may be most convenient for the integration.

In the next article we shall investigate a criterion to distinguish equations which have, from those which have not, a determinate value. The latter, though not derivable from a single equation between two variables, may be deduced from two equations between three variables by elimination.

45. *Required to find whether a formula of a higher order*

than the first, in which the fluxion of neither variable is constant, has a determinate value.

It follows from the nature of differentiation that in the equations $dy = pdx$, $dp = qdx$, $dq = rdx$, $\cdots$ the quantities p, q, r, $\cdots$ are solely dependent upon the form of the function $y = fx$, and that they are not affected by any supposition that may be made with respect to d^2x, or the higher orders of dx. From these equations the values of d^2y, d^3y, $\cdots$ may be obtained by differentiation; and if they be substituted in the proposed formula, it will, in general, contain p, q, r, $\cdots$ dx, d^2x, $\cdots$; of which the value of d^2x will be affected by the independent variable. But if the formula should be such that after these substitutions, and without making any supposition with respect to the independent variable, d^2x and the higher orders of dx should of themselves disappear, it contains only p, q, r, $\cdots$ and dx, and will therefore possess the same value whatever function be taken to be the independent variable.

Thus, making these substitutions in $\frac{yd^2x+xd^2y}{dx^2}$ and $\frac{dyd^2x-dxd^2y}{dx^3}$, since $d^2y = pd^2x + qdx^2$, they become $\frac{(y+px)d^2x}{dx^2} + xq$ and $-q$; of which the first is indeterminate, being dependent upon the hypothesis made with respect to d^2x; but the second possesses a determinate value.

Cor. The equation of condition that the canonical form of the second order, $Pd^2x + Qd^2y + Rdx^2 + Sdxdy + Td^2y$ may have a determinate value is $Pdx + Qdy = 0$.

For, since $d^2y = pd^2x + qdx^2$, the coefficient of d^2x becomes $P + Qp$ which is to $= 0$, or $Pdx + Qdy = 0$.

The equation $Pdx + Qdy = 0$ is not necessarily identical.

In the following article we shall give a few examples to illustrate the preceding propositions. It appears from the criterion that these examples possess a determinate value.

46. EXAMPLES.

Ex. 1. $Pdy^3 = dx^2dy - xdyd^2x + xdxd^2y$ where $P = Fx$.

1st. Suppose x to be the principal variable; then

$$Pdy^3 = dx^2dy + xdxd^2y, \text{ or } \frac{Pdy}{x^2} = \frac{dx^2dy + xdxd^2y}{x^2dy^2} \quad . \quad . \quad .$$

$= - d\frac{dx}{xdy}$ or $d\frac{dx}{xdy} = -\frac{\text{P}dy}{x^2}$; $\therefore \frac{dx}{xdy} \cdot d\frac{dx}{xdy} = -\frac{\text{P}dx}{x^3}$

$\therefore$ integrat., $\frac{1}{2}\cdot\frac{dx^2}{dy^2} = \frac{c}{2} - \int\frac{\text{P}dx}{x^3}$ and $dy = \frac{dx}{x\sqrt{c - 2\int\frac{\text{P}dx}{x^3}}}$.

2dly. Let dy be constant; then $\text{P}dy^2 = dx^2 - xd^2x \therefore \cdots$

$-\frac{\text{P}dy^2}{x^2} = \frac{xd^2x - dx^2}{x^2} = d\frac{dx}{x} \therefore$ . . . . . .

$\frac{dx}{x}\,d\,\frac{dx}{x} = -dy^2.\frac{\text{P}dx}{x^3} \therefore$ integrating, $\frac{1}{2}\frac{dx^2}{x^2} = \frac{cdy^2}{2} - dy^2\int\frac{\text{P}dx}{x^3}$

or $dy = \frac{dx}{\sqrt{c - 2\int\frac{\text{P}dx}{x^3}}}$.

3dly. Let xdy be constant; then, differentiating, $dxdy + xd^2y = 0$; and the proposed equation becomes $\text{P}dy^2 = -xd^2x$.

Suppose $xdy = a$; then $dy = \frac{a}{x} \therefore \frac{\text{P}a^2}{x^2} = -xd^2x \therefore \cdots$

$\frac{\text{P}a^2dx}{x^3} = -dxd^2x \therefore$ integrating,

$\frac{dx^2}{2} = ca^2 - a^2\int\frac{\text{P}dx}{x^3} = cx^2dy^2 - x^2dy^2\int\frac{\text{P}dx}{x^3}$ or

$dy = -\frac{dx}{x\sqrt{c - 2\int\frac{\text{P}dx}{x^3}}}$.

4thly. Let $\frac{dx}{x}$ be constant; then $xd^2x - dx^2 = 0$; and the proposed equation becomes $\text{P}dy^3 = xdxd^2y$.

Suppose $\frac{dx}{x} = a$ or $dx = ax \therefore \text{P}dy^3 = ax^2d^2y = \frac{a^2x^3d^2y}{dx}$

$\therefore \frac{\text{P}dx}{x^3} = \frac{a^2d^2y}{dy^3} \therefore$ integrating, $\frac{a^2}{2dy^2} = \frac{c}{2} - \int\frac{\text{P}dx}{x^3}$, or

$dy = -\frac{a}{\sqrt{c - 2\int\frac{\text{P}dx}{x^3}}} = \frac{dx}{x\sqrt{c - 2\int\frac{\text{P}^2dx}{x^3}}}$; a result in which

the variables are separated, and which is the same in all the four cases.

Ex. 2. $d^2ydx - dyd^2x - ydx^3 = 0$.

This equation has been obtained on the hypothesis that both dx and dy vary; and in order to integrate it, suppose x to be the independent variable, or $d^2x = 0$, and it becomes

$d^2ydx - ydx^3 = 0$ or $\frac{d^2y}{dx^2} - y = 0$; $\therefore$ integrating,

$x = l\frac{y + \sqrt{y^2 + c^2}}{c'}$; the same that would result if y were the independent variable.

Ex. 3. $xy(dxd^2y - dyd^2x) = ydydx^2 - y^2dQdy^2 - xdxdy^2$ where $Q = Fy$; or

$xydxd^2y + y^2dQdy^2 = xydyd^2x + ydydx^2 - xdxdy^2$.

If the right side of this equation be divided by y^2dy, it becomes the fluxion of $\frac{xdx}{y}$. Let then $\frac{xdx}{y} = a$ be taken for the constant; then the proposed becomes

$xydxd^2y + y^2dQdy^2 = 0$; or $dQ = -\frac{xdx}{y} \cdot \frac{d^2y}{dy^2} = -a \cdot \frac{d^2y}{dy^2}$;

$\therefore Q = \frac{a}{dy} + c = \frac{xdx}{ydy} + c$ $\therefore Qydy = xdx + cydy$, and the required primitive is $2\int Qydy = x^2 + cy^2 + c'$.

Ex. 4. $x^3d^2x + x^2yd^2y + (a^2 - y^2)dx^2 + x^2dy^2 = 0$.

Here $x^2(xd^2x + yd^2y) + (a^2 - y^2)dx^2 + x^2dy^2 = 0$ $\therefore$ assume $d(xdx + ydy) = 0$; and by substitution, we have $-x^2dx^2 + (a^2 - y^2)dx^2 = 0$, or $\sqrt{a^2 - y^2 - x^2} \cdot dx = 0$; and since dx cannot $= 0$, we have $a^2 - y^2 - x^2 = 0$, or $a^2 = x^2 + y^2$; so that this method in this example does not give the complete primitive. (Vid. Lacroix, tome i. p. 219).

Ex. 5. $xd^2x + yd^2y + dx^2 = 0$.

Assume $d(xdx + ydy) = 0$; then, by substitution, we have $-dy^2 = 0$, which is impossible; or there is no single relation between the variables which satisfies the proposed equation.

If both in this and the preceding example there be substituted $dy = pdx$, $dp = qdx$ according to the rule contained in art. 45, in each case, in order that the coefficient of d^2x may vanish, we must have $x + yp = 0$, or $xdx + ydy = 0$. Now this satisfies the first, but not the second; which

shows that the first has, and the second has not, a determinate value.

47. MISCELLANEOUS PRAXIS.

1. $\frac{d^2y}{dx^2} + a^2y = 0 \therefore y = c\sin.(ax + c')$.

2. $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \therefore c'y = x^2 - c^2$.

3. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \therefore y = c'l\frac{x}{c}$.

4. $\frac{dy}{dx} + (a + x)\frac{d^2y}{dx^2} = \frac{x}{b} \therefore y = \frac{1}{2b}\int\frac{(x^2+c^2)dx}{x+a}$.

5. $x\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 \therefore y = c\int e^{\int\frac{dx}{x}}dx - x + c'$.

6. $nxd^2y + \sqrt{dx^2 + dy^2} = 0 \therefore$ . . . . . .

$$y = \frac{ncx^{\frac{n-1}{n}}}{2(n-1)} - \frac{nx^{\frac{n+1}{n}}}{2c(n+1)} + c'.$$

7. $\frac{d^2y}{dy^2} = 2(ax + b) \therefore y = \int\frac{dx}{c - 2bx - ax^2}$.

8. $2\sqrt{a^2 - ax}\,dxd^2y = (dx^2 + dy^2)^{\frac{3}{2}}$ belongs to the cycloid.

9. $yd^2y + 2dy^2 = ydx^2 \therefore x = 5^{\frac{1}{2}}\int\frac{y^2dy}{\sqrt{c+2y^5}}$.

10. $dsd^2y = dx^3$ where ds = const. $\therefore$ . . . . . .

$$2y = ce^x + \frac{1}{ce^x} + c'.$$

11. $ds^2d^2y = xdx^4$ where ds = const. $\therefore p = \int xdx$.

12. $\frac{d^3y}{dx^3} - \frac{3}{1-x}\frac{d^2y}{dx^2} = 0 \therefore y = \frac{cx^2+c'x+c''}{1-x}$.

13. $yx^2d^2y + (ydx - xdy)^2 = 0 \therefore y^2 = cx + c'x^2$.

14. $xd^2y - yd^2x = d^2u$ is an exact fluxion, and $du = xdy - ydx$.

15. $\frac{d^ny}{dx^n} = y \therefore y = c'e^{a'x} + c''e^{a''x} \ldots + c^{(n)}e^{a^{(n)}x}$ where $a', a'' \ldots a^{(n)}$ are the n roots of unity.

16. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \therefore y = ce^x \sin.(x + c')$.

17. $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (y - m)x^2 = 0 \therefore$ . . . . . .

$y - m = ce^{x - \frac{x^2}{2}} + c'e^{-\left(x + \frac{x^2}{2}\right)}$.

18. $\frac{d^4y}{dx^4} - y = k \therefore y + k = ce^x + c'e^{-x} + c''\sin x + c'''\cos x$.

19. $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} + y = 0 \therefore$ . . . . . .

$y = ce^{\frac{x\sqrt{3}}{2}} \sin.\left(\frac{x}{2} + c'\right) + c''e^{\frac{-x\sqrt{3}}{2}} \sin.\left(\frac{x}{2} + c'''\right)$.

20. $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0 \therefore$ . . . .

$y = e^x(a + bx) + c\sin.(x + c')$.

21. $\frac{d^ny}{dx^n} - \frac{n}{1}\frac{d^{n-1}y}{dx^{n-1}} + \frac{n(n-1)}{1.2}\frac{d^{n-2}y}{dx^{n-2}} - \cdots - \frac{n}{1}\frac{dy}{dx} + y = 0$

$\therefore y = e^x(c + c'x + c''x^2 \cdots + c^{(n-1)}x^{n-1})$.

22. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4 \therefore y = cx + c'x^2 + 2$.

23. $\frac{d^2y}{dx^2} + \frac{y}{x^2} = ax^m + bx^n \therefore$ . . . . . .

$y = cx^{\frac{1+\sqrt{-3}}{2}} + c'x^{\frac{1-\sqrt{-3}}{2}} + \frac{ax^{m+2}}{m^2+3m+3} + \frac{bx^{n+2}}{n^2+3n+3}$.

24. $\frac{d^4y}{dz^4} - \frac{4}{z}\frac{d^3y}{dz^3} + \frac{12}{z^2}\frac{d^2y}{dz^2} - \frac{24}{z^3}\frac{dy}{dz} + \frac{24}{z^4}y = 0$, where $x = lz$

$\therefore y = ce^x + c'e^{2x} + c''e^{3x} + c'''e^{4x}$.

25. $xydyd^2x - xydxd^2y + ydydx^2 - xdxdy^2 = 0$, and

26. $xd^2x + yd^2y + dx^2 + dy^2 = 0$ have a determinate value.

27. $x^3d^2x + y^3d^2y + 6xydxdy = 0$ has not a determinate value.

CHAPTER VII.

Total Fluxional Equations of more than one Variable.

1. FLUXIONAL equations of three or more variables are either total or partial: they are called partial, when they contain the partial fluxional coefficients of one of the variables considered as a function of the others.

Total fluxional equations of the first order and degree between three variables are of the form $Pdx+Qdy+Rdz=0$ where P, Q and R are functions of (x, y, z).

If it be evident from inspection that the proposed equation is an exact fluxion, it is to be integrated by the common rules, and a correction must be added dependent upon the conditions of the question. Thus, of $yzdx+xzdy+xydz=0$, the fluent is $xyz=a^3$ where a can be determined, if any corresponding values of x, y, z be known.

If either of the variables can be separated from the other two, the equation is to be integrated by the rules for the integration of a formula of two variables. Thus, let the equation be reducible to the form $zdz=Mdx+Ndy$, where $z=fz$, and M and N are each functions of (x, y); then we have, by substitution, $du=Mdx+Ndy$, which is to be integrated as in ch. 4; and $u=\int zdz$ is to be found as in ch. 1.

2. All equations of two variables are, or are considered to be integrable; but this is not true of equations of three variables: it will be shown in the course of the chapter that there are equations of three variables which do not admit of integration; or at least, that they cannot be integrated without assuming a relation between the variables which is not included in the proposed equation. Equations of this kind are not *independently integrable.*

3. *Required to investigate when a formula of the first order and of three or more variables is an exact fluxion.*

Let the proposed formula be $Pdx+Qdy+Rdz=du$;

then, if du be an exact fluxion, we have $\frac{du}{dx} = \text{P}$, $\frac{du}{dy} = \text{Q}$ and $\frac{du}{dz} = \text{R}$; wherefore $\frac{d\text{P}}{dy} = \frac{d^2u}{dxdy} = \frac{d^2u}{dydx} = \frac{d\text{Q}}{dx}$ (1).

Similarly $\frac{d\text{P}}{dz} = \frac{d\text{R}}{dx}$ (2) and $\frac{d\text{Q}}{dz} = \frac{d\text{R}}{dy}$ (3).

Hence the proposed formula is an exact fluxion only when these three equations obtain.

If the formula contain n variables, there will be $\frac{n(n-1)}{2}$ equations of condition. (Alg. 228.)

Cor. 1. $\text{P}dx + \text{Q}dy$, $\text{P}dx + \text{R}dz$ and $\text{Q}dy + \text{R}dz$ are each exact fluxions; where z is considered constant in the first, y in the second and x in the third.

Cor. 2. Conversely, if each of the binomials is an exact fluxion, the whole formula is also an exact fluxion.

4. *Required to investigate when an equation of three variables is independently integrable.*

Let $du = \text{P}dx + \text{Q}dy + \text{R}dz = 0$ be the equation; whence $dz = -\frac{\text{P}}{\text{R}}dx - \frac{\text{Q}}{\text{R}}dy$; and by substitution, let this be $dz = pdx + qdy$ (α); where p and q are each functions of (x, y, z).

Now if there is any single equation which expresses the relation between x, y and z; *i. e.* if $z = \text{F}(x, y) = v$, we have $dz = \frac{dv}{dx}dx + \frac{dv}{dy}dy$; and this must be equivalent to, though not under the same form as (α); which shows that when $z = \text{F}(x, y)$, $pdx + qdy$ is an exact fluxion.

Applying Euler's criterion, we have $\frac{\dot{dp}}{dy} = \frac{\dot{dq}}{dx}$. But $\frac{\dot{dp}}{dy} = \frac{dp}{dy} + \frac{dp}{dz}\frac{dz}{dy} = \frac{dp}{dy} + q\frac{dp}{dz}$. Similarly $\frac{\dot{dq}}{dx} = \frac{dq}{dx} + p\frac{dq}{dz}$; and consequently, $\frac{dp}{dy} + q\frac{dp}{dz} = \frac{dq}{dx} + p\frac{dq}{dz}$,

or $\frac{dq}{dx} - \frac{dp}{dy} + p\frac{dq}{dz} - q\frac{dp}{dz} = 0.$ (β).

But $\frac{dq}{dx} = \frac{\mathrm{Q}\frac{d\mathrm{R}}{dx} - \mathrm{R}\frac{d\mathrm{Q}}{dx}}{\mathrm{R}^2}$ and $\frac{dp}{dy} = \frac{\mathrm{P}\frac{d\mathrm{R}}{dy} - \mathrm{R}\frac{d\mathrm{P}}{dy}}{\mathrm{R}^2}$; whence, by substitution, our equation of condition becomes

$$\mathrm{P}\left\{\frac{d\mathrm{Q}}{dz} - \frac{d\mathrm{R}}{dy}\right\} + \mathrm{Q}\left\{\frac{d\mathrm{R}}{dx} - \frac{d\mathrm{P}}{dz}\right\} + \mathrm{R}\left\{\frac{d\mathrm{P}}{dy} - \frac{d\mathrm{Q}}{dx}\right\} \quad (\gamma).$$

Cor. If the equation contain four variables, we shall have by substitution $dz = pdx + qdy + rdv$; and by a similar process we may obtain *three* equations of condition. Generally, if the number of variables be n, the number of equations of condition shall be $\frac{(n-1)(n-2)}{1.2}$.

5. It appears then from the preceding investigation, that in the equation $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz = 0$, we are not at liberty to assume that one of the variables as z is a function of the other two. If the equation of condition (β) or (γ) does not obtain, z is not a function of (x, y); or, in other words, there is no primitive which alone and independently expresses the relation between the three variables. It will be shown, however, in the course of the chapter that even in this case the variables bear a certain relation to each other which is not altogether arbitrary, but is subject to restrictions. In fact Monge has applied equations of this class with great success to the theory of curves of double curvature. (Mémoires de l'Académie des Sciences de Paris, ann. 1784.)

When the equation (γ) obtains, it does not necessarily follow that the *formula* $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz$ is an exact fluxion; but, since the *equation* $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz = 0$ is integrable, if it is not an exact fluxion, it must be capable of being rendered one by the introduction of a factor. We ought then to arrive at the same equation of condition, if we begin by inquiring when the formula $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz$ is capable of being rendered an exact fluxion by the introduction of a factor. This shall be verified in the next article.

6. *Required to find when a formula of the first order and of three variables may be rendered an exact fluxion by the introduction of a factor.*

Let $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz$ be the proposed formula; t, the requisite factor; then $t\mathrm{P}dx + t\mathrm{Q}dy + t\mathrm{R}dz = du$ is to be an exact fluxion; and consequently (art. 3), we have

$$\left.\begin{aligned} \mathrm{P}\frac{dt}{dy} - \mathrm{Q}\frac{dt}{dx} + t\left\{\frac{d\mathrm{P}}{dy} - \frac{d\mathrm{Q}}{dx}\right\} &= 0 \\ \mathrm{P}\frac{dt}{dz} - \mathrm{R}\frac{dt}{dx} + t\left\{\frac{d\mathrm{P}}{dz} - \frac{d\mathrm{R}}{dx}\right\} &= 0 \\ \mathrm{Q}\frac{dt}{dz} - \mathrm{R}\frac{dt}{dy} + t\left\{\frac{d\mathrm{Q}}{dz} - \frac{d\mathrm{R}}{dy}\right\} &= 0 \end{aligned}\right\};$$

and multiplying the first of these by R, the second by $-$ Q and the third by P, and adding the results, the factor t will disappear, and we shall obtain the same equation (γ) as in art. 4; which shows that unless this equation obtain, the proposed cannot be rendered an exact fluxion, whatever be the form of t.

7. We should now proceed to find a factor which renders the formula integrable per se. The process is the same as that adopted, ch. 4, art. 28; but the resulting equations, as in that article, are as difficult to integrate as the proposed formula. They will contain not only the variables and their fluxional coefficients, but also t and its fluxional coefficients $\frac{dt}{dx}$, $\frac{dt}{dy}$, $\frac{dt}{dz}$. We shall not stop to consider the cases in which t can be found. If the proposed be homogeneous, it may be shown as in ch. 4, 29, case 2, that

$$t = \frac{1}{\mathrm{P}x + \mathrm{Q}y + \mathrm{R}z}.$$

8. *Required to integrate a formula of three variables which is independently integrable.*

The rule is the same as that of ch. 4, 24, rule 2, where there are only two variables: 'First integrate on the supposition that one of the variables as z is constant, in which case the correction is Z a sole function of z; then differentiate the result partially with respect to z, by which the value of Z is found and the required fluent obtained.'

It may be sometimes convenient to begin by integrating on the supposition that two of the variables are constant; in which case the correction is an implicit function of those variables; and we must then differentiate the result with respect to both of them.

In the following examples it is supposed that the test has been applied, and that they are independently integrable. If the formula is not independently integrable, we shall not be able to express the arbitrary function in terms of its own variable solely.

9. EXAMPLES.

Ex. 1. $du = yzdx + xzdy + xydz.$

The equations of condition (art. 3) obtain; and integrating on the supposition that z is constant, $u = z\,xy + z$ $\therefore$ $\frac{du}{dz} = xy + \frac{dz}{dz} = xy$ $\therefore$ $dz = 0$ $\therefore$ $z = c$ and $u = xyz + c.$

Ex. 2. $du = (y + z)dx + (x + z)dy + (x + y)dz.$

The equations of condition (art. 3) obtain; and if $dz = 0$, $du = ydx + xdy + z(dx + dy)$ $\therefore$ $u = xy + z(x + y) + z$ $\therefore$ $\frac{du}{dz} = x + y + \frac{dz}{dz} = x + y$ $\therefore$ $dz = 0$ and $u = xy + xz + yz.$

Ex. 3. $du = \frac{ydy + xdx + zdz}{\sqrt{y^2 + x^2 + z^2}} + \frac{zdx - xdz}{x^2 + z^2} + zdz.$

First integrate on the supposition that x and z are constant; then the correction $= \mathrm{F}(x, z) = \mathrm{xz}$; $\therefore$ . . . . $u = \sqrt{y^2 + x^2 + z^2} + \mathrm{xz}$; and differentiating with respect to x and z, $du = \frac{xdx + zdz}{\sqrt{y^2 + x^2 + z^2}} + d.\mathrm{xz} = \frac{xdx + zdz}{\sqrt{y^2 + x^2 + z^2}} \cdots$

$+ \frac{zdx - xdz}{x^2 + z^2} + zdz$ $\therefore$ $\mathrm{xz} = \tan.^{-1}\frac{x}{z} + \frac{z^2}{2} + c$ $\therefore$ . . .

$u = \sqrt{y^2 + x^2 + z^2} + \tan.^{-1}\frac{x}{z} + \frac{z^2}{2} + c.$

Ex. 4. $du = \frac{1}{y^3}(xydz + 2z^2dy - 2xzdy + yzdx - 2yzdz).$

Here $du = \frac{zdx}{y^2} - \frac{2zxdy}{y^3} + \frac{2z^2dy}{y^3} + \frac{x - 2z}{y^2}dz$ $\therefore$ . . . .

$u = \frac{zx - z^2}{y^2} + z$ $\therefore$ $\frac{du}{dz} = \frac{x - 2z}{y^2} + \frac{dz}{dz} = \frac{x - 2z}{y^2}$ $\therefore$ $\frac{dz}{dz} = 0$ $\therefore$

$z = c$ and $u = \frac{xz - z^2}{y^2} + c.$

Ex. 5. $\frac{y + z}{y}dx + \frac{y^2 - xz}{y^2}dy - \frac{x + y}{z}dz = 0 = du.$

It appears from art. 3 that this is not an exact fluxion; but that its two first members form an exact fluxion, whose fluent is $x + y + \frac{xz}{y}$. It also appears from equation (β) that it is independently integrable; wherefore

$u = x + y + \frac{xz}{y} + z = 0 \therefore \frac{du}{dz} = \frac{x}{y} + \frac{dz}{dz} = -\frac{x+y}{z}$, or

$-\frac{dz}{dz} = \frac{x+y}{z} + \frac{x}{y} = -\frac{z}{z} \therefore \frac{dz}{z} = \frac{dz}{z}$, or $z = cz$, and the required primitive is $xy + xz + y^2 = cyz$.

Ex. 6. $(y^2 + yz + z^2)dx + (x^2 + xz + z^2)dy \ . \ . \ .$
$+ (x^2 + xy + y^2)dz = 0.$

Suppose z constant; then $\frac{dx}{x^2+xz+z^2} + \frac{dy}{y^2+yz+z^2} = 0$;

and, integrating,

$\frac{2}{2\sqrt{3}}\text{tan.}^{-1}\frac{x\sqrt{3}}{x+2z} + \frac{2}{2\sqrt{3}}\text{tan.}^{-1}\frac{y\sqrt{3}}{y+2z} = \frac{2}{2\sqrt{3}}\text{tan.}^{-1}z \therefore \ . \ . \ .$

$z = \frac{xz+yz+xy}{2z^2+xz+yz-xy}$.

By differentiating this totally, Lacroix obtains a value of z in terms of z; but as the process is long and uninstructive, we shall integrate this example by a shorter method.

The equation is symmetrical with respect to all the variables; if then we can find the relation between two of them, the third being supposed of any determinate magnitude whatever, the required primitive will be obtained.

Suppose $z = 0$; then the equation becomes

$\frac{dx}{x^2} + \frac{dy}{y^2} = 0$, and, integrating,

$\frac{1}{x} + \frac{1}{y} = \frac{1}{c}$, or $(x + y)c = xy$; and the required primitive is $(x + y + z)c = xy + xz + yz$.

By this method we might have integrated ex. 1 and 2, and all formulæ in which the variables are symmetrical.

10. *Required to find when an homogeneous equation of three variables and of the first order is independently integrable.*

Let $\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz = 0$ be an homogeneous equation of n dimensions. Substitute $x = sz$ and $y = tz$; then $\mathrm{P} = \mathrm{S}z^n$, $\mathrm{Q} = \mathrm{T}z^n$, and $\mathrm{R} = \mathrm{V}z^n$; where S, T and V are implicit functions of s and t not containing z.

Since $dx = sdz + zds$ and $dy = tdz + zdt$, the equation becomes, dividing by z^n, $\mathrm{S}zds + \mathrm{T}zdt + (s\mathrm{S} + t\mathrm{T} + \mathrm{V})dz = 0$, or $-\frac{dz}{z} = \frac{\mathrm{S}ds + \mathrm{T}dt}{s\mathrm{S} + t\mathrm{T} + \mathrm{V}} =$, by substitution, $\sigma ds + \tau dt$; which therefore is to be an exact fluxion, and consequently the condition is that $\frac{d\sigma}{dt} = \frac{d\tau}{ds}$, where $\sigma = \frac{\mathrm{S}}{s\mathrm{S} + t\mathrm{T} + \mathrm{V}}$ and $\tau = \frac{\mathrm{T}}{s\mathrm{S} + t\mathrm{T} + \mathrm{V}}$.

Cor. If this condition be satisfied, the fluent is

$$l\frac{c}{z} = \int \frac{\mathrm{S}ds + \mathrm{T}dt}{s\mathrm{S} + t\mathrm{T} + \mathrm{V}}.$$

The examples of the preceding article may be integrated by this formula.

Ex. 1. $du = xydz + 2z^2dy - 2xzdy + yzdx - 2yzdz = 0$, or $yzdx + (2z^2 - 2xz)dy + (xy - 2yz)dz = 0$.

Substitute $x = sz$, $y = tz$ ∴ . . . . . $t(sdz + zds) + 2(1 - s)(tdz + zdt) + (ts - 2t)dz = 0$, or $tzds + 2zdt - 2szdt = 0$, or $\frac{ds}{1-s} + \frac{2dt}{t} = 0$ ∴ integrating, $t^2 = c(1 - s)$, or $\frac{y^2}{z^2} = c\left(1 - \frac{x}{z}\right)$ ∴ $y^2 = c(z^2 - xz)$.

Ex. 2. $(x^2 - y^2 + z^2)dx - z^2dy + z(y - x)dz$ . . . $+ \frac{x(y^2 - x^2)}{z}dz = 0$.

Substitute $x = sz$, $y = tz$; then $(s^2 - t^2 + 1)(sdz + zds) - (tdz + zdt) + (t - s)dz$ . . . $+ s(t^2 - s^2)dz = 0$, or $(s^2 - t^2 + 1)ds - dt = 0$.

There is no direct method of integrating this equation; for it is not one of those forms which are integrable by Riccati's transformation. But, observing that $t = s$ is a solution, and applying Euler's test (ch. 5, 19), it appears that $t = s$ is a particular and not a singular solution.

Substitute then $t = s + v$; and the equation becomes

$-\frac{dv}{v^2}-\frac{2sds}{v}=ds$; and integrating, we have

$$\frac{1}{v}=e^{s^2}\left\{c+\int e^{-s^2}ds\right\} \text{ or } \frac{e^{-s^2}}{t-s}=\int e^{-s^2}ds+c.$$

This method of deducing the complete primitive from a particular solution is applicable to all equations of the form $dy+y^2dx=\text{x}dx$.

11. If the proposed equation is of a higher degree than the first, the preceding methods fail; and we must search for some other criterion of integrability. Now it follows from the rules of differentiation established, vol. i. ch. 1, that the first fluxion of any function whatever of three or more variables must be of the form $\text{M}dx+\text{N}dy+\text{P}dz+\cdots$; if therefore the proposed equation be solved with respect to dz, which we will suppose of the second degree, when it is independently integrable, the result must be an equation in which dx and dy are not under a radical sign; which furnishes a new condition of integrability.

Having then solved the quadratick with respect to dz, the equation is not independently integrable, unless the result consists of two equations each of the form $dz=mdx+ndy$; and applying to these the equation of condition (γ), it will be seen whether the required primitive consists of one or of two equations.

The primitive, when it is a single equation, is the product of the primitives of the two equations $dz=mdx+ndy$.

Ex. $\text{P}dx^2+\text{Q}dy^2+\text{R}dz^2+\text{S}dxdy+\text{T}dxdz+\text{V}dydz=0$.

If this equation be solved with respect to dz, the quantity under the radical sign is

$(\text{T}^2-4\text{PR})dx^2+2(\text{TV}-\text{RS})dxdy+(\text{V}^2-4\text{QR})dy^2$; and that this may disappear as a radical, we must have

$(\text{T}^2-4\text{PR})(\text{V}^2-4\text{QR})=(\text{TV}-\text{RS})^2$. Unless this equation obtains, the primitive is not independently integrable.

12. *Total fluxional equations of the second order.*

Equations of this kind seldom occur. They appear under a form in which the first fluxion of one or more of the variables is constant. We have considered the formula of two variables in which dx is constant in ch. 6, 17.

Next let the formula be $\text{R}dxdy+\text{S}dy^2+\text{T}dx^2=d^2u$ where both dx and dy are constant.

(1). To find the condition that d^2u may be an exact

fluxion, let it be the fluxion of $\text{M}dx + \text{N}dy$; then the term $\text{R}dxdy$ is the sum of two terms; the one, arising from the differentiation of $\text{N}dy$ with respect to x; the other, from the differentiation of $\text{M}dx$ with respect to y. Let then $\text{R} = \text{R}' + \text{R}''$, and d^2u becomes $(\text{T}dx + \text{R}'dy)dx + (\text{R}''dx + \text{S}dy)dy$.

This is an exact fluxion when $\dfrac{d(\text{T}dx + \text{R}'dy)}{dy} = \dfrac{d(\text{R}''dx + \text{S}dy)}{dx}$

or $\dfrac{d\text{T}}{dy}dx + \dfrac{d\text{R}'}{dy}dy = \dfrac{d\text{R}''}{dx}dx + \dfrac{d\text{S}}{dx}dy$. But x and y, and consequently dx and dy, are independent quantities; wherefore we have $\dfrac{d\text{T}}{dy} = \dfrac{d\text{R}''}{dx}$ and $\dfrac{d\text{R}'}{dy} = \dfrac{d\text{S}}{dx}$.

To eliminate R' and R''; we have $\text{R}'' = \text{R} - \text{R}'$, and consequently $\dfrac{d\text{R}''}{dx} = \dfrac{d\text{R}}{dx} - \dfrac{d\text{R}'}{dx}$; and by substitution,

$\dfrac{d\text{T}}{dy} = \dfrac{d\text{R}}{dx} - \dfrac{d\text{R}'}{dx}$, or $\dfrac{d\text{R}'}{dx} = \dfrac{d\text{R}}{dx} - \dfrac{d\text{T}}{dy}$ (1).

Also $\dfrac{d\text{R}'}{dy} = \dfrac{d\text{S}}{dx}$ (2); wherefore, differentiating the first with respect to y, and the second with respect to x, there results $\dfrac{d^2\text{S}}{dx^2} + \dfrac{d^2\text{T}}{dy^2} = \dfrac{d^2\text{R}}{dxdy}$; which is therefore the equation of condition that d^2u may be an exact fluxion.

(2). Required to integrate d^2u when it is an exact fluxion.

Let $du = \text{M}dx + \text{N}dy$; then $\text{T}dx^2$ is the fluxion of $\text{M}dx$ and $\text{S}dy^2$ the fluxion of $\text{N}dy$; wherefore
$du = dx\int \text{T}dx + dy\int \text{S}dy$ where y is constant in $\int \text{T}dx$ and x in $\int \text{S}dy$.

In correcting the fluent, we must introduce at the first integration two arbitrary constants of the form $adx + bdy$; for if the complete primitive contain $ax + by$, both a and b will be eliminated by the second differentiation. Similarly, if a primitive contain $ax^2 + by^2$, both the constants will be eliminated by the third differentiation; and so on.

(3). The equation of condition deduced in (1) may be verified by the result obtained in (2).

For, differentiating, $du = \text{M}dx + \text{N}dy$, and comparing the result with the proposed formula, we obtain

$\text{R} = \frac{d\text{M}}{dy} + \frac{d\text{N}}{dx}$, or $\frac{d\text{N}}{dx} = \text{R} - \frac{d\text{M}}{dy}$. Also $\frac{d\text{N}}{dy} = \text{S}$; whence $\frac{d\text{S}}{dx} = \frac{d^2\text{N}}{dydx} = \frac{d^2\text{N}}{dxdy} = \frac{d\text{R}}{dy} - \frac{d^2\text{M}}{dy^2} = \frac{d\text{R}}{dy} - \frac{d^2fr dx}{dy^2}$. And again differentiating with respect to x, there results $\frac{d^2\text{S}}{dx^2} = \frac{d^2\text{R}}{dydx} - \frac{d^2\text{T}}{dy^2}$ which is the equation of condition deduced in (1).

Ex. $d^2u = y^2dx^2 + 4xydxdy + x^2dy^2$.

Here the equation of condition obtains, and

$du = y^2xdx + x^2ydy + adx + bdy$ and $u = \frac{x^2y^2}{2} + ax + by + c$.

(4). Next let the formula contain *three* variables.

If z is a function of (x, y); *i. e.* if the proposed formula, considered as an equation, is independently integrable; we have, differentiating on the hypothesis that both dx and dy are constant,

$dz = pdx + qdy$
$d^2z = rdx^2 + 2sdxdy + tdy^2$ } where p, q, r, s and t are the fluxional coefficients of $z = \text{F}(x, y)$. (Vol. i. ch. 4, 18.)

If dx and dy both vary; then

$d^2z = rdx^2 + 2sdxdy + tdy^2 + pd^2x + qd^2y$; which, by making d^2x or $d^2y = 0$, is adapted to the hypothesis of dx or of dy being the only constant.

Now it may be shown, as in the case of two variables (ch. 6, 45), that when the proposed equation is independently integrable, if these values of dz and d^2z be substituted in it, the quantities d^2x and d^2y will not be found in the result, which will contain only the variables and their fluxional coefficients. This therefore furnishes a criterion by which we can ascertain whether an equation of three variables and of the second order be independently integrable.

Ex. $(xdx + zdz)d^2y - zdyd^2z - dy(dx^2 + dy^2 + dz^2) = 0$.

Here $d^2z = rdx^2 + 2sdxdy + tdy^2 + qd^2y$; and by substituting for dz and d^2z, the equation becomes

$$\left.\begin{array}{l}(x + zp)dxd^2y - zdy(rdx^2 + 2sdxdy + tdy^2) \\ - dy\{(1 + p^2)dx^2 + 2pqdxdy + (1 + q^2)dy^2\}\end{array}\right\} = 0.$$

This cannot be independently integrable unless $x + zp = 0$. Integrating, we have $x^2 + z^2 = 2\text{Y}$: and differentiating in order to find Y, $xdx + zdz = d\text{Y}$; and again differen-

tiating on the proposed hypothesis, viz. that dx is constant, we have $dx^2 + dz^2 + zd^2z = d^2\mathrm{Y}$; and by substitution, the equation becomes $d\mathrm{Y}d^2y - dyd^2\mathrm{Y} - dy^3 = 0$, or

$\frac{dyd^2\mathrm{Y} - d\mathrm{Y}d^2y}{dy^2} = -dy$; $\therefore$ integrating, $\frac{d\mathrm{Y}}{dy} = a - y$ and

$\mathrm{Y} = b + ay - \frac{y^2}{2}$, and consequently $x^2+y^2+z^2=2ay+2b$;

which is the complete primitive of the proposed equation.

13. *Required to investigate the equations of condition that a fluxional function of any order and of any number of variables may be an exact fluxion*

Let the proposed function be of the nth order; then, if the fluxion of one of the variables as x be supposed constant, it may be put under the form $\mathrm{v}dx^m$, where m depends upon the degree of the function, and v is a function of $(x, y, z, \cdots)$ and of the fluxional coefficients as far as the nth order.

Substitute $\left.\begin{array}{l} dy = pdx,\ dp = qdx, \cdots ds = tdx \\ dz = p'dx,\ dp' = q'dx, \cdots ds' = t'dx \\ \cdots = \cdots \end{array}\right\}$ then

v is a function of $x, y, z \cdots; p, q \cdots s, t; p', q' \cdots s', t'$; &c.

The first fluent of $\mathrm{v}dx^m$ is of the $n-1$th order. Let it be udx^{m-1}; then $du = \mathrm{v}dx$, where $u = \mathrm{F}(x, y, z \cdots p, q \cdots s; p', q' \cdots s'$; &c.).

$$\text{Now } \mathrm{v}dx = du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dp}dp \cdots + \frac{du}{ds}ds + \frac{du}{dz}dz + \frac{du}{dp'}dp' \cdots + \frac{du}{ds'}ds', + \cdots$$

$$\text{or } \mathrm{v} = \frac{du}{dx} = \frac{du}{dx} + \frac{du}{dy}p + \frac{du}{dp}q \cdots + \frac{du}{ds}t + \frac{du}{dz}p' + \frac{du}{dp'}q' \cdots + \frac{du}{ds'}t' + \cdots$$

$$\text{Let } d\mathrm{v} = \mathrm{M}dx + \mathrm{N}dy + \mathrm{P}dp + \mathrm{Q}dq \cdots + \mathrm{T}dt + \mathrm{N}'dz + \mathrm{P}'dp' + \mathrm{Q}'dq' \cdots + \mathrm{T}'dt' + \cdots$$

Then we have

$$\text{N} = \frac{d\text{V}}{dy} = \frac{1}{dx}\, d\frac{du}{dy}; \text{ for } \text{V} = \frac{1}{dx} du$$

$$\text{P} = \frac{d\text{V}}{dp} = \frac{d^2u}{dxdp} + \frac{du}{dy} + \frac{d^2u}{dydp}p + \frac{d^2u}{dp^2}q \cdots + \frac{d^2u}{dsdp}t$$
$$+ \frac{d^2u}{dzdp}p' + \frac{d^2u}{dp'dp}q' \cdots + \frac{d^2u}{ds'dp}t'$$
$$+ \cdots$$
$$= \frac{du}{dy} + \frac{1}{dx}\, d\frac{du}{dp}.$$

Similarly

$$\text{Q} = \frac{d\text{V}}{dq} = \frac{du}{dp} + \frac{1}{dx}\, d\frac{du}{dq}$$
$$\cdots = \cdots$$
$$\text{T} = \frac{d\text{V}}{dt} = \frac{du}{ds}.$$

If the proposed function is of the *first* order, u does not contain p; and we have $\text{P} = \frac{du}{dy}$; wherefore N, which

$= \frac{1}{dx}\, d\, \frac{du}{dy}$, $= \frac{1}{dx} d\text{P}$, or $\text{N} - \frac{\dot{d}\text{P}}{dx} = 0$.

If the function is of the *second* order, $\text{P} = \frac{du}{dy} + \frac{1}{dx} d\text{Q}$;

wherefore N, $= \frac{1}{dx}\, d\frac{du}{dy}$, $= \frac{\dot{d}\text{P}}{dx} - \frac{\dot{d}^2\text{Q}}{dx^2}$, or $\text{N} - \frac{\dot{d}\text{P}}{dx} + \frac{\dot{d}^2\text{Q}}{dx^2} = 0$.

Generally, an equation of condition that a function of the nth order may be an exact fluxion is

$$\text{N} - \frac{\dot{d}\text{P}}{dx} + \frac{\dot{d}^2\text{Q}}{dx^2} - \frac{\dot{d}^3\text{R}}{dx^3} + \cdots = 0, \text{ where } \text{N} = \frac{d\text{V}}{dy}, \text{P} = \frac{d\text{V}}{dp}, \cdots$$
$$\text{Q} = \frac{d\text{V}}{dq}, \cdots$$

A similar equation may be deduced for any other of the variables as z; for $\text{N}' = \frac{1}{dx}\, d\frac{du}{dz}$; $\text{P}' = \frac{du}{dz} + \frac{1}{dx}\, d\frac{du}{dp'}$;

$\text{Q}' = \frac{du}{dp'} + \frac{1}{dx}\, d\frac{du}{dq'}$; $\cdots$ and it may be shown that an equa-

tion of condition is $\text{N}' - \frac{d\text{P}'}{dx} + \frac{d^2\text{Q}'}{dx^2} - \frac{d^3\text{R}'}{dx^3} + \cdots = 0.$

Hence it appears that the number of equations of condition that a function of the nth order of any number of variables may be an exact fluxion is equal to the number of the variables exclusive of the independent.

Next to find the equations of condition that udx^{m-1} may be an exact fluxion.

Let $du' = udx$, where u' is of the $n - 2$th order; then it has been shown that the equation of condition corresponding to the variable y is

$$\frac{du}{dy} - \frac{1}{dx}d\frac{du}{dp} + \frac{1}{dx^2}d^2\frac{du}{dq} - \cdots = 0.$$

But, since $\text{P} = \frac{du}{dy} + \frac{1}{dx}d\frac{du}{dp}$, we have

$$\frac{du}{dy} = \text{P} - \frac{1}{dx}\left\{d\text{Q} - \frac{1}{dx}d^2\frac{du}{dq}\right\} = \text{P} - \frac{d\text{Q}}{dx} + \frac{d^2\text{R}}{dx^2} - \frac{d^3\text{S}}{dx^3} + \cdots$$

$$\frac{du}{dp} = \text{Q} - \frac{1}{dx}\left\{d\text{R} - \frac{1}{dx}d^2\frac{du}{dr}\right\} = \text{Q} - \frac{d\text{R}}{dx} + \frac{d^2\text{S}}{dx^2} - \cdots$$

$$\cdots = \cdots$$

wherefore by substitution the equation becomes

$$\text{P} - \frac{2d\text{Q}}{dx} + \frac{3d^2\text{R}}{dx^2} - \frac{4d^3\text{S}}{dx^3} + \cdots = 0.$$

In the same manner by supposing $du'' = u'dx$, where u'' is of the $n - 3$d order, it may be shown that the equation of condition, corresponding to the variable y, in order that $u'dx^{m-2}$ may be an exact fluxion is $\text{Q} - \frac{3d\text{R}}{dx} + \frac{6d^2\text{S}}{dx^2} - \frac{10d^3\text{T}}{dx^3} + \cdots$

Hence collecting all the equations of condition corresponding to the variable y, they are

$$\text{N} - \frac{1}{dx}d\text{P} + \frac{1}{dx^2}d^2\text{Q} - \frac{1}{dx^3}d^3\text{R} + \frac{1}{dx^4}d^4\text{S} - \cdots = 0$$

$$\text{P} - \frac{2}{dx}d\text{Q} + \frac{3}{dx^2}d^2\text{R} - \frac{4}{dx^3}d^3\text{S} + \cdots = 0$$

$$\text{Q} - \frac{3}{dx}d\text{R} + \frac{6}{dx^2}d^2\text{S} - \cdots = 0$$

$$\text{R} - \frac{4}{dx}d\text{S} + \cdots = 0$$

$\cdots$ &c.

Similarly there may be deduced n similar equations corresponding to any other of the variables, the independent excepted.

Cor. In the article it is supposed that the function has an independent variable: if it has not, assume t to be the independent variable which connects all the others; then the proposed function is of the form $\text{V}dt^m$.

Adopting Lagrange's notation, let $\frac{dx}{dt} = x'$, $\frac{d^2x}{dt^2} = x''$, $\cdots$ $\frac{dy}{dt} = y'$, $\frac{d^2y}{dt^2} = y''$ $\cdots$ &c.; then assuming

$$\left.\begin{aligned} d\text{V} = \text{M}dt + \text{N}dx + \text{P}dx' + \text{Q}dx'' + \cdots \\ + \text{P}'dy' + \text{Q}'dy'' + \cdots \\ + \cdots \end{aligned}\right\}, \text{ it may be shown}$$

as in the article that the equation of condition corresponding to x that the function may be an exact fluxion is

$$\text{N} - \frac{1}{dt}d\text{P} + \frac{1}{dt^2}d^2\text{Q} - \cdots = 0.$$

$$\left.\begin{aligned} \text{But } \frac{1}{dt}d\text{P} = \frac{1}{dt}\,d\frac{d\text{V}}{dx}, = d\frac{d\text{V}}{dx_1}, \text{ if } x_1 = dx \\ \frac{1}{dt^2}d^2\text{Q} = \frac{1}{dt^2}d^2\frac{d\text{V}}{dx''} = d^2\frac{d\text{V}}{dx_2}, \text{ if } x_2 = d^2x \\ \cdots = \cdots \end{aligned}\right\}; \text{ from which}$$

it appears that the equations may be expressed independently of t in the following more convenient form:

$$\left.\begin{aligned} \frac{d\text{V}}{dx} - d\frac{d\text{V}}{dx_1} + d^2\frac{d\text{V}}{dx_2} - \cdots = 0 \\ \frac{d\text{V}}{dy} - d\frac{d\text{V}}{dy_1} + d^2\frac{d\text{V}}{dy_2} - \cdots = 0 \\ \cdots \end{aligned}\right\} \text{where } \begin{aligned} x_1 = dx, x_2 = d^2x, \cdots \\ y_1 = dy, y_2 = d^2y, \cdots \\ \cdots \end{aligned}$$

Similarly we may express the equations of condition that the proposed function may be an nth exact fluxion.

Ex. 1. $\text{A}dx + \text{B}dy$ where A and B each $= \text{F}(x, y)$.

Here $du = \text{V}dx = (\text{A} + \text{B}p)dx$; $\therefore$ N, which

$= \frac{dV}{dy}$, $= \frac{dA}{dy} + \frac{dB}{dy}p$.

Also $P = B$; $\therefore \frac{\dot{d}P}{dx} = \frac{dB}{dx} + \frac{dB}{dy}p$, and by the equation of condition, $\frac{dA}{dy} + \frac{dB}{dy}p = \frac{dB}{dx} + \frac{dB}{dy}p$, or $\frac{dA}{dy} = \frac{dB}{dx}$.

Ex. 2. $Adx + Bdy + Cdz = (A + Bp + Cp')dx$.

Here $V = A + Bp + Cp'$

$$\left.\begin{aligned} N &= \frac{dA}{dy} + \frac{dB}{dy}p + \frac{dC}{dy}p'; \; P = B \\ N' &= \frac{dA}{dz} + \frac{dB}{dz}p + \frac{dC}{dz}p'; \; P' = C \end{aligned}\right\}.$$

The two equations of condition are $N - \frac{\dot{d}P}{dx} = 0$;

$$N' - \frac{\dot{d}P'}{dx} = 0.$$

From the first,

$\frac{dA}{dy} + \frac{dB}{dy}p + \frac{dC}{dy}p' = \frac{\dot{d}B}{dx} = \frac{dB}{dx} + \frac{dB}{dy}p + \frac{dB}{dz}p'$; or

$\frac{dA}{dy} - \frac{dB}{dx} + \left\{\frac{dC}{dy} - \frac{dB}{dz}\right\}p' = 0 \therefore$ (Alg. 347), $\frac{dA}{dy} - \frac{dB}{dx} = 0$

and $\frac{dC}{dy} - \frac{dB}{dz} = 0$, or $\frac{dA}{dy} = \frac{dB}{dx}$ and $\frac{dB}{dz} = \frac{dC}{dy}$.

From the second,

$\frac{dA}{dz} - \frac{dC}{dx} + \left\{\frac{dB}{dz} - \frac{dC}{dy}\right\}p = 0$, or $\frac{dA}{dz} = \frac{dC}{dx}$ and $\frac{dB}{dz} = \frac{dC}{dy}$.

Ex. 3. $Ad^2y + Bdx^2$, where A and B are functions of (x, y, p).

Here $V = Aq + B \therefore$

$N = \frac{dA}{dy}q + \frac{dB}{dy}$

$P = \frac{dA}{dp}q + \frac{dB}{dp}$ and $Q = A \therefore$

$$\left.\begin{aligned}\frac{\dot{d}\mathrm{P}}{\dot{d}x} &= \frac{d^2\mathrm{A}}{dpdx}q + \frac{d^2\mathrm{A}}{dpdy}qp + \frac{d^2\mathrm{A}}{dp^2}q^2 + \frac{d\mathrm{A}}{dp}r \\ &+ \frac{d^2\mathrm{B}}{dpdx} + \frac{d^2\mathrm{B}}{dpdy}p + \frac{d^2\mathrm{B}}{dp^2}q\end{aligned}\right\}$$

$$\frac{\dot{d}\mathrm{Q}}{\dot{d}x} = \frac{d\mathrm{A}}{dx} + \frac{d\mathrm{A}}{dy}p + \frac{d\mathrm{A}}{dp}q \therefore$$

$$\left.\begin{aligned}\frac{\dot{d}^2\mathrm{Q}}{\dot{d}x^2} &= \frac{d^2\mathrm{A}}{dx^2} + \frac{d^2\mathrm{A}}{dxdy}2p + \frac{d^2\mathrm{A}}{dxdp}2q \\ &+ \frac{d^2\mathrm{A}}{dy^2}p^2 + \frac{d^2\mathrm{A}}{dydp}2pq + \frac{d\mathrm{A}}{dy}q \\ &+ \frac{d^2\mathrm{A}}{dp^2}q^2 + \frac{d\mathrm{A}}{dp}r.\end{aligned}\right\}$$

Substituting these in the equation of condition, there will result

$$\left.\begin{aligned}&\frac{d\mathrm{B}}{dy} - \frac{d^2\mathrm{B}}{dpdx} - \frac{d^2\mathrm{B}}{dpdy}p + \frac{d^2\mathrm{A}}{dx^2} + \frac{d^2\mathrm{A}}{dxdy}2p + \frac{d^2\mathrm{A}}{dy^2}p^2 \\ &+ \left\{\frac{2d\mathrm{A}}{dy} + \frac{d^2\mathrm{A}}{dpdx} + \frac{d^2\mathrm{A}}{dydp}p - \frac{d^2\mathrm{B}}{dp^2}\right\}q\end{aligned}\right\} = 0;$$

which, since A and B do not contain q, may be separated into the two

$$\frac{2d\mathrm{A}}{dy} + \frac{d^2\mathrm{A}}{dxdp} + \frac{d^2\mathrm{A}}{dydp}p - \frac{d^2\mathrm{B}}{dp^2} = 0\ (\alpha) \text{ and}$$

$$\frac{d\mathrm{B}}{dy} - \frac{d^2\mathrm{B}}{dxdp} - \frac{d^2\mathrm{B}}{dydp}p + \frac{d^2\mathrm{A}}{dx^2} + \frac{d^2\mathrm{A}}{dxdy}2p + \frac{d^2\mathrm{A}}{dy^2}p^2 = 0\ (\beta),$$

which are the conditions in order that the proposed may be an exact fluxion.

When these conditions obtain, since $du = \mathrm{v}dx = \mathrm{A}dp + \mathrm{B}dx$, we have by partial integration, $u = \int \mathrm{A}dp + v$, where v is a function of (x, y) not containing p. Vid. ch. 8, 4, cor. Differentiating this with respect to x and y, and comparing it with the proposed, we find the value of $\frac{dv}{dx}dx + \frac{dv}{dy}dy$, and by integration obtain the value of v and consequently of u.

If it be further required that the proposed be a *second* exact fluxion, another equation of condition must obtain, viz. $\mathrm{P} - \frac{2}{dx}d\mathrm{Q} = 0$, or $\frac{d\mathrm{B}}{dp} - \frac{2d\mathrm{A}}{dx} - \frac{2d\mathrm{A}}{dy}p - \frac{d\mathrm{A}}{dp}q = 0$;

whence $\frac{d\mathrm{A}}{dp} = 0$ (γ), or A may not contain p.

The three equations (α), (β) and (γ) must obtain in order that the proposed may be a second exact fluxion.

Thus $(2xydy + x^2ydx)d^2y + xdy^3 + (y + x^2)dy^2dx + (2 + 3y)xydydx^2 + y^3dx^3 = \mathrm{v}dx^3$ satisfies (α) and (β) but not (γ); and we have $u = \int \mathrm{A}dp + v = \int (2xyp + x^2y)dp + v = xyp^2 + x^2yp + v$. But $u = \int \mathrm{v}dx^2$; $\therefore$ differentiating partially with respect to x and y, there results

$$\frac{dv}{dx}dx + \frac{dv}{dy}dy + p^2d.xy + pd.x^2y = (xp^3 + (x^2 + y)p^2 + (2 + 3y)xyp + y^3)dx, \text{ or } \frac{dv}{dx}dx + \frac{dv}{dy}dy = (3xy^2p + y^3)dx$$

$= y^3dx + 3xy^2dy$ $\therefore \mathrm{v} = xy^3 + c$ and $u = xyp^2 + x^2yp + xy^3 + c$; and the required fluent, which $= udx^2$,

$= xydy^2 + x^2ydydx + xy^3dx^2 + cdx^2$.

Ex. 4. $yd^2x - xd^2y = \mathrm{v}$.

Here $\left.\begin{array}{l} \frac{d\mathrm{v}}{dx} = -d^2y\,;\ \frac{d\mathrm{v}}{dx_1} = 0\,;\ \frac{d\mathrm{v}}{dx_2} = y \therefore d^2\frac{d\mathrm{v}}{dx_2} = d^2y \\ \frac{d\mathrm{v}}{dy} = d^2x;\ \frac{d\mathrm{v}}{dy_1} = 0\,;\ \frac{d\mathrm{v}}{dy_2} = -x \therefore d^2\frac{d\mathrm{v}}{dy_2} = -d^2x \end{array}\right\}$

$\therefore$ v is an exact fluxion and $u = ydx - xdy$.

Ex. 5. $\mathrm{E}d^2x + \mathrm{A}d^2y + \mathrm{B}dy^2 + \mathrm{c}dxdy + \mathrm{D}dx^2 = 0$, which is the general linear equation of the second order.

Here it may be shown that the equations of condition are

$$\mathrm{D} = \frac{d\mathrm{E}}{dx},\ \mathrm{B} = \frac{d\mathrm{A}}{dy} \text{ and } \mathrm{c} = \frac{d\mathrm{A}}{dx} + \frac{d\mathrm{E}}{dy}.$$

This result agrees with ch. 6, art. 17; for if dx be constant, $d^2x = 0$, or the first term vanishes; and to eliminate E, we have $\frac{d\mathrm{c}}{dx} = \frac{d^2\mathrm{A}}{dx^2} + \frac{d^2\mathrm{E}}{dydx} = \frac{d^2\mathrm{A}}{dx^2} + \frac{d\mathrm{D}}{dy}$ (β), and

$\mathrm{B} = \frac{d\mathrm{A}}{dy}$ (α) as before.

If dy be also constant, $d^2y = 0$; and to eliminate A, we have from (β) $\frac{d^3\mathrm{A}}{dx^2dy} = \frac{d^2\mathrm{c}}{dxdy} - \frac{d^2\mathrm{D}}{dy^2}$. But from ($\alpha$),

$\frac{d^3A}{dydx^2} = \frac{d^2B}{dx^2}$; whence $\frac{d^2B}{dx^2} + \frac{d^2D}{dy^2} = \frac{d^2C}{dxdy}$, which agrees with art. 12.

14. We should now investigate the conditions that an equation of any order and of any number of variables may be *independently integrable*. The student will find no difficulty in deducing these from the equations of the last article by the same method as the equation (γ) is deduced in art. 6.

Having ascertained that an equation is independently integrable, the next step in the process of integration is to find the factor which will render it an exact fluxion; but it may be observed here as in ch. 4, art. 28, that the equations which arise for determining the factor are in general more difficult to integrate than the proposed.

One of the cases which Euler has considered is that of $d^2y + \mathrm{R}dxdy + \mathrm{s}dy^2 + \mathrm{T}dx^2 = 0$, where R, S and T and the requisite factor, are functions of (x, y) which do not contain dy or dx.

If z be the factor; it may be shown on this hypothesis that $\frac{dz}{z} = \mathrm{Q}dx + \mathrm{s}dy$, or $z = e^{\int(\mathrm{Q}dx + \mathrm{s}dy)}$, where

$$\mathrm{Q} = \frac{\frac{d^2\mathrm{R}}{dydx} - \frac{d^2\mathrm{S}}{dx^2} - \frac{d^2\mathrm{T}}{dy^2} - \frac{d.\mathrm{ST}}{dy} - \mathrm{R}\frac{d\mathrm{s}}{dx}}{\frac{2d\mathrm{s}}{dx} - \frac{d\mathrm{R}}{dy}}.$$

When z has been determined, the proposed may be integrated, and it may be shown that the required first primitive is $zdy + cdx + dx\int\{\mathrm{T}dx + (\mathrm{R} - \mathrm{Q})dy\} = 0$.

The conditions that the proposed case may be independently integrable are that $\mathrm{Q}dx + \mathrm{s}dy$ and $\mathrm{T}dx + (\mathrm{R} - \mathrm{Q})dy$ are both exact fluxions.

From the first we have $\frac{d\mathrm{Q}}{dy} = \frac{d\mathrm{s}}{dx}$ (1). From the second it may be shown that $\frac{d(\mathrm{R}-\mathrm{Q})}{dx} - \frac{d\mathrm{T}}{dy} = \mathrm{ST} - \mathrm{Q}(\mathrm{R} - \mathrm{Q})$ (2).

If $\frac{2d\mathrm{s}}{dx} - \frac{d\mathrm{R}}{dy} = 0$, the value of z fails. In this case, substitute $\mathrm{Q} = \frac{\mathrm{R}}{2} + \mathrm{Q}'$; then, it may be shown that Q' is a sole function of x.

Also from equation (2), $\frac{d\left(\frac{\mathrm{R}}{2}-\mathrm{Q}'\right)}{dx}-\frac{d\mathrm{T}}{dy}=\mathrm{ST}-\frac{\mathrm{R}^2}{4}+\mathrm{Q}'^2$,

or $\frac{d\mathrm{Q}'}{dx}+\mathrm{Q}'^2=\frac{1}{2}\frac{d\mathrm{R}}{dx}-\frac{d\mathrm{T}}{dy}+\frac{\mathrm{R}^2}{4}-\mathrm{ST}$; which therefore is a sole function of x. Since this is an equation of the first order between two variables Q' and x, it may be integrated and Q' obtained in terms of x, and thus z may be found, which $= e^{\int\left(\frac{\mathrm{R}}{2}+\mathrm{Q}'\right)dx+\mathrm{S}dy)}$.

The linear equation $\frac{d^2y}{dx^2}+\frac{\mathrm{R}dy}{dx}+\mathrm{T}=0$, where $\mathrm{T}=\mathrm{S}y-\mathrm{X}$, is an instance of this case; but we shall not apply the method here pointed out, since the proposed equation has been already exhibited by a shorter process.

As examples of the above formulæ, which we have not demonstrated on account of the length of the calculation, the student may take the following.

Ex. 1. $d^2y+\frac{x^3+2y}{xy}dydx+\frac{2dy^2}{y}+2xdx^2=0$.

This is independently integrable; $z=x^2y^2$; and the required first primitive is $2x^2y^2dy+cdx+x^4y^2dx=0$.

Ex. 2. $d^2y+\frac{1-4x+x^2y}{x}dydx+\left\{\mathrm{X}-\frac{4y}{x}+y^2\right\}dx^2\cdots$ $=0$.

Here $\mathrm{Q}=\frac{1}{x}$; $z=x$; and the first primitive is

$2xdy+(c+x^2y^2-8xy+2\int\mathrm{X}xdx)dx=0$.

Euler has also integrated several equations of the second order by means of factors of given forms which render them exact fluxions as in ch. 4, art. 33 and 34; but we shall entirely omit these, and shall conclude this part of the subject with integrating the general linear equation of two variables.

15. *Required to integrate the linear equation of the nth order by means of a factor.*

Let the equation be $\frac{d^ny}{dx^n}+\mathrm{P}\frac{d^{n-1}y}{dx^{n-1}}+\mathrm{Q}\frac{d^{n-2}y}{dx^{n-2}}\cdots+\mathrm{T}y=\mathrm{X}$;

and let zdx be that factor which renders both sides of this equation an exact fluxion, z being a function of x.

Since $z\left\{\frac{d^ny}{dx^n} + \mathrm{P}\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{Q}\frac{d^{n-2}y}{dx^{n-2}} \cdots + \mathrm{T}y\right\}dx$ is an exact fluxion, we have (art. 13)

$$\frac{d^nz}{dx^n} - \frac{1}{dx^{n-1}}d^{n-1}\cdot z\mathrm{P} + \frac{1}{dx^{n-2}}d^{n-2}\cdot z\mathrm{Q} - \cdots \pm z\mathrm{T} = 0\ (\alpha).$$

If one of the conjugate values of this equation be known, one of the first primitives of the proposed may be found.

For the first primitive is of the form

$$z\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{P}'\frac{d^{n-2}y}{dx^{n-2}} + \mathrm{Q}'\frac{d^{n-3}y}{dx^{n-3}} \cdots + \mathrm{T}'y = \int z\mathrm{X}dx + c\ (\beta);$$

and to find P', Q', $\cdots$ T', differentiate this and equate its terms with the corresponding terms of the proposed; and there results

$$\mathrm{P}' = z\mathrm{P} - \frac{dz}{dx}$$

$$\mathrm{Q}' = z\mathrm{Q} - \frac{d\mathrm{P}'}{dx} = z\mathrm{Q} - \frac{d.z\mathrm{P}}{dx} + \frac{d^2z}{dx^2}$$

$$\mathrm{R}' = z\mathrm{R} - \frac{d.z\mathrm{Q}}{dx} + \frac{d^2.z\mathrm{P}}{dx^2} - \frac{d^3z}{dx^3}$$

$$\cdots = \cdots$$

$$\mathrm{T}' = z\mathrm{T} - \frac{d.z\mathrm{S}}{dx} + \cdots \pm \frac{d^{n-2}.z\mathrm{P}}{dx^{n-2}} \mp \frac{d^{n-1}z}{dx^{n-1}}.$$

Similarly if the n values of (α) are known, its n first primitives and consequently the complete primitive may be found.

If only $n - 1$ values of (α) are known, we may eliminate all the fluxional coefficients except $\frac{dy}{dx}$, in which case there will result an equation of the form $\frac{dy}{dx} + \mathrm{M}y = \mathrm{X}'$; from which the complete primitive may be obtained.

The complete primitive may be also obtained, if $n - 1$ conjugate values of the proposed are known when $\mathrm{X} = 0$.

For let y' be one of them; then, substituting it in (β), it becomes $z\frac{d^{n-1}y'}{dx^{n-1}} + \mathrm{P}'\frac{d^{n-2}y'}{dx^{n-2}} + \mathrm{Q}'\frac{d^{n-3}y'}{dx^{n-3}} \cdots + \mathrm{T}y' = c$ where

P', Q', $\cdots$ T' contain the $n-1$ fluxional coefficients of z. Similarly there may be obtained $n-1$ equations, each containing a different arbitrary constant; and if all the $n-1$ fluxional coefficients of z be eliminated from these except the first, there will result a linear equation of the first order, from which z may be found. This value of z contains n constants, and consequently we may obtain from it n conjugate values of z; and, as before, the complete fluent of the proposed equation may be found.

Ex. $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{y}{x^2}=\dfrac{a}{x^2-1}$.

Let $z\left\{\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{y}{x^2}\right\}dx$ be an exact fluxion; then (art. 13) $\dfrac{d^2z}{dx^2}-\dfrac{1}{dx}d.\dfrac{z}{x}-\dfrac{z}{x^2}=0$, or $\dfrac{d^2z}{dx^2}-\dfrac{1}{dx}\dfrac{dz}{x}=0$, or $\dfrac{d^2z}{dz}=\dfrac{dx}{x}$ $\therefore$ integrating, $dz=cxdx$ and $z=cx^2+c'$ $\therefore$ $z'=1$ and $z''=x^2$.

Substituting these values in (β), the two first primitives of the proposed are $\dfrac{dy}{dx}+\dfrac{y}{x}=\int\dfrac{adx}{x^2-1}+c$ and $x^2\dfrac{dy}{dx}=\int\dfrac{ax^2dx}{x^2-1}+c'$; $\therefore$ the required fluent is

$$xy=cx^2-c'+x^2\int\frac{adx}{x^2-1}-\int\frac{ax^2dx}{x^2-1}.$$

16. When the equation (γ) of art. 4, does not obtain, the proposed equation neither is nor can be rendered an exact fluxion; and, as has been already observed, the required primitive cannot be an equation, in which one of the variables as z is a function of the other two x and y considered as independent quantities; but this hinders not that it may be a function of x and y connected by a second independent equation; in which case the required primitive shall consist of *two* independent equations. We shall consider then in the next article, whether by establishing a new relation between the variables, we are enabled to find two equations which by differentiation and elimination will satisfy the proposed.

In these cases the proposed is not *independently integrable;* and it is manifest that the new relation which is to

be introduced, as it does not form part of the data of the question, must be to a certain degree arbitrary; or that the primitive contains arbitrary and indeterminate functions.

17. *Required to integrate an equation of three variables, which is not independently integrable.*

Let $mdx + ndy + pdz = 0$ be the equation, where m, n and p are functions of (x, y, z); and let t be that factor which renders the two first terms an exact fluxion, z being considered constant; then, if it rendered the whole function an exact fluxion, we should have
$du = t(mdx + ndy + pdz) = 0$ an exact fluxion.

Substitute $dv = t(mdx + ndy)$ and $R = tp$; then the equation becomes $du = dv + Rdz = 0$ (1); and integrating on the supposition that z is constant, $u = v + \varphi z = 0$ (2); and differentiating partially, $\frac{du}{dz} = \frac{dv}{dz} + \frac{d\varphi z}{dz}$. But, were du an exact fluxion, we should have $\frac{du}{dz} = R$,

or $R = \frac{dv}{dz} + \frac{d\varphi z}{dz}$ (3); from which φz may be obtained, and substituting its value in (2), there would result the required primitive as in art. 8.

But (ex hyp.) $du = dv + Rdz$ is not an exact fluxion, and consequently it is not one of the data that $\frac{du}{dz} = R$, or $\frac{dv}{dz} + \frac{d\varphi z}{dz} = R$; wherefore, the new relation which must be introduced in order that the proposed equation may be satisfied is that $\frac{dv}{dz} + \frac{d\varphi z}{dz} = R$.

Now making this assumption, we have the two equations $v + \varphi z = 0$ (γ) and $\frac{dv}{dz} + \frac{d\varphi z}{dz} = R$ (γ); and differentiating the first of these totally, since $t(mdx + ndy)$ is the fluxion of v when differentiated partially with respect to x and y, we have $t(mdx + ndy) + \frac{dv}{dz}dz + d\varphi z = 0$; but from the second, $\frac{dv}{dz}dz + d\varphi z = Rdz$; whence, by substitution, there re-

sults $t(mdx + ndy) + \text{R}dz = 0$, the proposed equation; or the required primitive consists of the two equations

$$\left.\begin{aligned} \text{V} + \varphi z &= 0 \\ \frac{d\text{V}}{dz} + \varphi' z &= \text{R} \end{aligned}\right\} (\gamma),$$ where φ denotes an arbitrary function.

18. The function φ in the preceding article resembles the arbitrary constant in common fluxional equations in this respect, that when the proposed equation is applied to the solution of problems, it ceases to be arbitrary, and its form is, in general, to be determined from the conditions of the question.

In the following examples, we suppose that it has been ascertained by means of equation (γ) art. 4, that they are not independently integrable.

19. EXAMPLES.

Ex. 1. $dz = zdx + ydy$.

Here $2zdx + 2ydy$ is an exact fluxion, if z be constant; or $\text{V} = 2zx + y^2$ and the two equations (γ), which compose the primitive, are

$$\left.\begin{aligned} \varphi z &= y^2 + 2xz \\ \varphi' z &= 2x + z \end{aligned}\right\}.$$

If the first be differentiated, and $d\varphi z$ be eliminated, there will result the proposed equation.

Ex. 2. $ydz + zdx + xdy = 0$.

Here $d\text{V} = \dfrac{zdx + xdy}{x}$, or $\text{V} = zlx + y$; and the required primitive is

$$\left.\begin{aligned} \varphi z &= zlx + y \\ \varphi' z &= lx - \frac{y}{x} \end{aligned}\right\}.$$

Ex. 3. $\dfrac{dz}{z-c} = \dfrac{xdx + ydy}{x(x-a) + y(y-b)}$.

Here $\text{V} = \dfrac{x^2+y^2}{2}$ and the primitive is

$$\left.\begin{aligned} \varphi z &= \frac{x^2+y^2}{2} \\ \varphi' z &= \frac{x(x-a)+y(y-b)}{z-c} \end{aligned}\right\}.$$

If $a = b = 0$, the example is independently integrable, and the primitive is $c(z - c) = x^2 + y^2$.

Ex. 4. $xdv + ydx + vdy + zdz = 0$.

Substitute $v - x = t$; then

$xdx + ydx + xdy + zdz + tdy + xdt = 0$ ∴ . . . . .

$x^2 + z^2 + 2vy = \varphi(v - x)$ }

$2y - 2x = \varphi'(v - x)$ } .

20. It may happen that by assigning a particular form to the arbitrary function, the two equations which compose the primitive become identical. In this case a single equation is obtained for the primitive; and it is a particular fluent, since it does not contain an arbitrary constant.

Thus the primitive of $(y - z)dz + xdy + (z - y)dx = 0$ is $\varphi z = \frac{z-y}{x}$ and $\varphi' z = \frac{y-z+x}{x^2}$. Suppose $\varphi z = 1$; then $\varphi' z = 0$; and each of these equations becomes $z = x + y$, which is therefore a particular fluent.

21. Since the primitive consists of two equations, and that there are three variables, if one of them as z be eliminated, there must result an equation between x and y; or y is an arbitrary function of x. Assuming then $y = \varphi x$, and eliminating y or x from the proposed, there will result an equation between two variables, the primitive of which combined with $y = \varphi x$ forms the required primitive. This, which appears the most direct method, is, in general, attended with great difficulty in consequence of the arbitrary function which is introduced into the equation to be integrated.

Ex. 1. $dz = aydx + bdy$.

Assume $y = \varphi' x$; then $dz = a\varphi' x dx + b\varphi'' x dx$; and integrating, $z = a\varphi x + b\varphi' x = a\varphi x + by$ which combined with $y = \varphi' x$ forms the required primitive.

Ex. 2. $\frac{dz}{z-c} = \frac{xdx+ydy}{x(x-a)+y(y-b)}$.

To integrate this by the same method, substitute $y = \varphi x$; then $\frac{dz}{z-c} = \frac{(x+\varphi x\varphi' x)dx}{x(x-a)+\varphi x(\varphi x-b)}$.

Since φ denotes an arbitrary function, we may assume $\varphi x = x$, or $y = x$, and there results

$\frac{dz}{z-c} = \frac{2xdx}{x(x-a)+x(x-b)} = \frac{2dx}{2x-a-b}$, or

$z-c = c(2x - a - b)$; or a primitive is

$$\left.\begin{aligned} y &= x \\ z - c &= c(2x - a - b) \end{aligned}\right\}.$$

This result is not included in the form obtained, art. 19, ex. 3; which shows that an equation which is not independently integrable may admit of other solutions than that which is given by the formula of art. 17.

We shall not consider equations which contain four variables; but refer the reader to the works of Monge and Paoli, or to the second volume of Lacroix's Calc. Integral.

22. *Equations of a higher degree than the first.*

The criterion of integrability of equations of the second degree has been given, art. 11. If it should appear that the equation is not independently integrable, we must integrate the proposed on the hypothesis that one of the variables is constant, and correct by an arbitrary function of that variable. Differentiating the result and equating it with the proposed, when raised to the same degree, there will arise a second equation which, together with the first, constitute the required primitive.

Ex. 1. $dz^2 = a^2(dx^2 + dy^2)$.

If x be constant, $dz = ady$; and integrating, $z = ay + \phi x$.

Differentiating this, $dz = ady + \phi' x dx$, or $dz^2 = a^2dy^2 + 2a\phi' x dx dy + \phi' x^2 dx^2$, which $\therefore = a^2(dx^2 + dy^2)$,

or $2dy = \dfrac{adx}{\phi' x} - \dfrac{1}{a}\phi' x dx$; and integrating,

$2y = a\int\dfrac{dx}{\phi' x} - \dfrac{1}{a}\phi x$; and the required primitive is

$$\left.\begin{aligned} z &= ay + \phi x \\ 2y &= a\int\frac{dx}{\phi' x} - \frac{1}{a}\phi x \end{aligned}\right\}.$$

Ex. 2. $z^2(dx^2 + dy^2 + dz^2) = a^2(dx^2 + dy^2)$.

If x be constant, $z^2(dy^2 + dz^2) = a^2dy^2$ or $dy = \dfrac{zdz}{\sqrt{a^2 - z^2}}$. $\therefore$

integrating, $y = \varphi x - \sqrt{a^2 - z^2}$.

Differentiating this, $dy = \phi' x dx + \dfrac{zdz}{\sqrt{a^2 - z^2}}$, or

$z^2dz^2 = (a^2 - z^2)(dy - \phi' x dx)^2$, which $\therefore = (a^2 - z^2)(dx^2 + dy^2)$,

or $2dy = \varphi' x dx - \frac{dx}{\varphi' x}$; and integrating, $2y = \varphi x - \int \frac{dx}{\varphi' x}$, or

$y = \sqrt{a^2 - z^2} - \int \frac{dx}{\varphi' x}$ ∴ the primitive is

$$\left.\begin{aligned} y &= \varphi x - \sqrt{a^2 - z^2} \\ y &= \sqrt{a^2 - z^2} - \int \frac{px}{\varphi' x} \end{aligned}\right\}.$$

To find the singular solutions, we must search among the radicals of the fluxional coefficients. In this case they contain $\sqrt{a^2 - z^2}$; and it appears that $a^2 - z^2 = 0$ is a singular solution.

23. *Equations of a higher order than the first.*

Let the equation be of the second order; and of the form $\mathrm{M}d^2x + \mathrm{N}d^2y + \mathrm{P}dx^2 = \mathrm{Q}dz^2$; where dz is constant, and M, N, P and Q are functions of x, y, z, dx, dy and dz.

Let t be the factor which renders the left side of this equation, considered as a function of two variables x and y, an exact fluxion $= d\mathrm{v}$. Then, integrating the proposed when multiplied by t, there results $\mathrm{v} = \varphi' z dz^m$, where m depends upon the *degree* of the equation.

Differentiating v on the supposition that every quantity in it varies except dz, which is our hypothesis, its whole fluxion is represented by

$\frac{d\mathrm{v}}{d^2x}d^2x + \frac{d\mathrm{v}}{d^2y}d^2y + \frac{d\mathrm{v}}{dx}dx + \frac{d\mathrm{v}}{dy}dy + \frac{d\mathrm{v}}{dz}dz$, which therefore $= \varphi'' z dz^{m+1}$. But comparing this with the proposed equation, we have the part

$\frac{d\mathrm{v}}{d^2x}d^2x + \frac{d\mathrm{v}}{d^2y}d^2y + \frac{d\mathrm{v}}{dx}dx + \frac{d\mathrm{v}}{dy}dy = t(\mathrm{M}d^2x + \mathrm{N}d^2y + \mathrm{P}dx^2)$,

which $= t\mathrm{Q}dz^2$; wherefore $t\mathrm{Q}dz^2 + \frac{d\mathrm{v}}{dz}dz = \varphi'' z dz^{m+1}$; and the two equations which constitute the first primitive are

$$\left.\begin{aligned} \mathrm{v} &= \varphi' z dz^m \\ \frac{d\mathrm{v}}{dz} &= \varphi'' z dz^m - t\mathrm{Q}dz \end{aligned}\right\}.$$

Ex. 1. $zdzd^2x - 2dyd^2y = zdz^3$.

Here $\mathrm{v} = zdzdx - dy^2$ and the first primitive is

$zdzdx - dy^2 = \varphi' z dz^2$
$zdz + dx = \varphi'' z dz$ }.

Ex. 2. $xzd^2x + zyd^2y + zdx^2 + zdy^2 + xydxdz + y^2dydz + z(1 + y)dz^2 = 0$.

Dividing by z, $dv = xd^2x + yd^2y + dx^2 + dy^2$; and integrating, $xdx + ydy = \varphi' z dz$ (1).

Differentiating this totally (dz excepted) and comparing it with the proposed, we have, dividing by dz,
$z\varphi'' z dz + xydx + y^2dy + z(1 + y)dz = 0$ (2), which together with (1) constitute the first primitive of the proposed equation.

Integrating the first primitive, there results $x^2 + y^2 = 2\varphi z$; and substituting in (2) $xdx + ydy = \varphi' z dz$, and dividing by dz, we have $z\varphi'' x + y\varphi' z + z(1 + y) = 0$; or the required primitive is $x^2 + y^2 = 2\varphi z$
$z(\varphi'' z + 1) + y(\varphi' z + z) = 0$ }.

To apply this equation; suppose that it results from the conditions of the question that $\varphi' z + z = 0$; then $\varphi'' z + 1 = 0$ and $\varphi z = \frac{c^2 - z^2}{2}$, and the equation is reduced to $x^2 + y^2 + z^2 = c^2$, which is therefore a particular primitive.

24. *On the application to geometry of equations which are not independently integrable.*

It appears from vol. i. ch. 7, arts. 57, 59, that a *curve of double curvature* or a curve in space is defined by the equations of two lines which are its orthographic projections upon two of the co-ordinate planes, as xz, yz. If we suppose two cylinders to be formed whose bases are these lines, it is manifest that their intersection forms the curve of double curvature; or a curve of double curvature may be considered as belonging to two cylindrical surfaces, the bases of which are in the planes xz, yz.

A *curve surface* is defined by the equation between x, y, z, the rectangular co-ordinates of any point in it; and either of these is a function of the other two considered as independent variables; and consequently any fluxional equation of three variables, which is independently integrable, whose primitive is of the form $F(x, y, z) = 0$, or $z = f(x, y)$ belongs to a curve surface; but if it be not independently integrable, its primitive consists of two independent equations; or it belongs to a curve of double curvature which may be considered as being situated in two different surfaces.

It further appears from this view of the subject that lines *in plano* form part of the geometry of two dimensions, but that lines in space, whether straight or curved, are included in the geometry of *three* dimensions.

25. SIMULTANEOUS EQUATIONS.

A system of equations between three or more variables is said to be integrated when its integration has been made to depend upon that of a single equation.

26. It appears from the nature of equations that if there are n different equations which contain $n + 1$ variables, any one of the variables may be considered as the principal, and all the others as functions of that one. By the process of elimination each variable becomes an implicit function of the principal; and solving the equation the function becomes explicit.

27. In this article we shall suppose that the system consists of *two* equations containing *three* variables; but the same observations are applicable to any system in which the number of the variables exceeds by unity the number of the equations.

Let $u = 0$, $v = 0$ be simultaneous equations, the one of the mth and the other of the nth order; and let them contain x, y, t, of which t is the principal variable.

Then $u = 0$ may contain t, dt, $d\cdot t$, $\cdots d^m t$; and $v = 0$ may contain t, dt, $d\cdot t$, $\cdots d^n t$; and since there are only two equations, we have not the means of eliminating these quantities; but differentiating the first n times and the second m times, we have upon the whole $m + n + 2$ equations which may contain t, dt, d^2t, $\cdots d^{m+n}t$, the number of which is $m + n + 1$; consequently an equation may be obtained by elimination which shall contain only x and y.

The process of elimination in these fluxional equations being difficult and tedious, this method has been superseded by D'Alembert's, which we shall give in the following articles. (Mémoires de Berlin, an. 1748.)

28. *Required to integrate a system of* two *linear equations of the first order and degree containing* three *variables.*

The canonical form, t being the principal variable, is

$$\left.\begin{aligned} \mathrm{M}\frac{dx}{dt} + \mathrm{N}\frac{dy}{dt} + \mathrm{R}x + \mathrm{S}y &= \mathrm{W} \\ \mathrm{M}'\frac{dx}{dt} + \mathrm{N}'\frac{dy}{dt} + \mathrm{R}'x + \mathrm{S}'y &= \mathrm{W}' \end{aligned}\right\}, \text{ or}$$

$$\left.\begin{aligned} \text{M}dx + \text{N}dy + (\text{R}x + \text{S}y)dt &= \text{W}dt \\ \text{M}'dx + \text{N}'dy + (\text{R}'x + \text{S}'y)dt &= \text{W}'dt \end{aligned}\right\}.$$

By eliminating dy and dx separately and by reduction, these may take the more simple form,

$$\left.\begin{aligned} dx + (\text{P}x + \text{Q}y)dt &= \text{T}dt \\ dy + (\text{P}'x + \text{Q}'y)dt &= \text{T}'dt \end{aligned}\right\}.$$

Multiplying the second by θ a function of t and adding it to the first, there results

$$dx + \theta dy + \{(\text{P} + \text{P}'\theta)x + (\text{Q} + \text{Q}'\theta)y\}dt = (\text{T} + \text{T}'\theta)dt.$$

Substitute $z = x + \theta y$; and the equation becomes, eliminating x,

$$dz - yd\theta + \{(\text{P} + \text{P}'\theta)(z - \theta y) + (\text{Q} + \text{Q}'\theta)y\}dt = (\text{T} + \text{T}'\theta)dt, \text{ or}$$

$$dz + (\text{P} + \text{P}'\theta)zdt - y\{d\theta + ((\text{P} + \text{P}'\theta)\theta - (\text{Q} + \text{Q}'\theta))dt\} \quad \cdot \quad \cdot \quad \cdot$$
$$= (\text{T} + \text{T}'\theta)dt.$$

Since θ is undetermined, we may assume

$d\theta + ((\text{P} + \text{P}'\theta)\theta - (\text{Q} + \text{Q}'\theta))dt = 0$ (α), which gives

$dz + (\text{P} + \text{P}'\theta)zdt = (\text{T} + \text{T}'\theta)dt$ (β).

Assigning then any value to θ which satisfies the equation (α) and substituting it in (β), (β) becomes a linear equation and consequently z which $= x + \theta y$ may be found; from which, if θ be eliminated, there will result the required primitive between x, y and t.

29. To take the case which occurs the most frequently; let P, Q, P′ and Q′ be constant quantities, T and T′ being as before functions of t.

Here equation (α) is satisfied by $d\theta = 0$, or θ = constant, and $(\text{P} + \text{P}'\theta)\theta - (\text{Q} + \text{Q}'\theta) = 0$, or $\theta^2 + \frac{\text{P} - \text{Q}'}{\text{P}'}\theta - \frac{\text{Q}}{\text{P}'} = 0$.

Let the values of θ deduced from this quadratick be θ and θ'; and substituting them in (β), let the corresponding values of $\text{P} + \text{P}'\theta$ and of $\text{T} + \text{T}'\theta$ be m, m' and V, V′; then integrating, and replacing z by its value $x + \theta y$, we have

$$\left.\begin{aligned} x + \theta y &= e^{-mt}\{\int e^{mt}\text{V}dt + c\} \\ x + \theta' y &= e^{-m't}\{\int e^{m't}\text{V}'dt + c'\} \end{aligned}\right\}, \text{ which therefore constitute}$$

the required primitive; and by elimination, x and y may be each obtained in terms of t.

The proposed system in this case belongs to two curve surfaces which intersect; and the resulting equations between x and t, y and t belong each to a curve line which is the projection of the intersection on two of the co-ordinate planes.

30. *Required to integrate a system of* three *equations of the first order and degree containing* four *variables.*

Let the form be reduced to

$$\left.\begin{aligned} dx + (\mathrm{P}x + \mathrm{Q}y + \mathrm{R}z)dt &= \mathrm{T}dt \\ dy + (\mathrm{P}'x + \mathrm{Q}'y + \mathrm{R}'z)dt &= \mathrm{T}'dt \\ dz + (\mathrm{P}''x + \mathrm{Q}''y + \mathrm{R}''z)dt &= \mathrm{T}''dt \end{aligned}\right\}.$$

Multiplying the second by θ and the third by φ, and adding the products to the first, there results

$$dx + \theta dy + \varphi dz$$
$$+\{(\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\phi)x + (\mathrm{Q} + \mathrm{Q}'\theta + \mathrm{Q}''\varphi)y + (\mathrm{R} + \mathrm{R}'\theta + \mathrm{R}''\varphi)z\}dt$$
$$= (\mathrm{T} + \mathrm{T}'\theta + \mathrm{T}''\varphi)dt.$$

Substitute $v = x + \theta y + \varphi z$; and the equation becomes, eliminating x, $dv + (\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\varphi)vdt$

$$- y\{d\theta + ((\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\varphi)\theta - (\mathrm{Q} + \mathrm{Q}'\theta + \mathrm{Q}''\varphi))dt\}$$
$$- z\{d\phi + ((\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\varphi)\phi - (\mathrm{R} + \mathrm{R}'\theta + \mathrm{R}''\varphi))dt\}$$
$$= (\mathrm{T} + \mathrm{T}'\theta + \mathrm{T}''\varphi)dt.$$

Assuming the coefficients of y and of z, each to $= 0$, there will result an equation linear in v, and also two equations from which θ and φ are to be found in terms of t.

Taking the case in which all the coefficients in the proposed form, except T, T′ and T″ are constant, the two equations for finding θ and ϕ are satisfied by

$$\left.\begin{aligned} d\theta &= 0 \\ d\phi &= 0 \end{aligned}\right\} \text{ and } \left.\begin{aligned} (\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\phi)\theta - (\mathrm{Q} + \mathrm{Q}'\theta + \mathrm{Q}''\phi) &= 0 \\ (\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\phi)\phi - (\mathrm{R} + \mathrm{R}'\theta + \mathrm{R}''\phi) &= 0 \end{aligned}\right\}.$$

It appears from the process of elimination that the equation for determining θ and ϕ, each rise to three dimensions. Let then the values of θ and ϕ be θ, θ', θ'' and ϕ, ϕ', ϕ''; and let the corresponding values of $\mathrm{P} + \mathrm{P}'\theta + \mathrm{P}''\phi$ and of $\mathrm{T} + \mathrm{T}'\theta + \mathrm{T}''\phi$ be m, m', m'' and V, V′, V″; then integrating the linear equation $dv + mvdt = \mathrm{V}dt$ and replacing V by its value $x + \theta y + \phi z$, there results

$$\left.\begin{aligned} x + \theta y + \phi z &= e^{-mt}\{\int e^{mt}\mathrm{V}dt + c\} \\ x + \theta' y + \phi' z &= e^{-m't}\{\int e^{m't}\mathrm{V}'dt + c'\} \\ x + \theta'' y + \phi'' z &= e^{-m''t}\{\int e^{m''t}\mathrm{V}''dt + c''\} \end{aligned}\right\} \text{ from which } x, y \text{ and } z \text{ may be each obtained in terms of } t.$$

The same method is manifestly applicable to any system of the first order and degree in which the number of the variables exceeds by unity the number of the equations.

31. *Required to integrate a linear system of any order.*

The proposed system is always reducible by substitution to one of the next inferior order in which the number of the

variables exceeds by unity the number of the equations; consequently by continued reduction the integration of the proposed may be made to depend upon a system of the first order; and it is therefore integrable.

Thus, let the proposed be under the form

$$\left.\begin{aligned} d^2x + (\mathrm{P}dx + \mathrm{Q}dy)dt + (\mathrm{R}x + \mathrm{S}y)dt^2 &= \mathrm{T}dt^2 \\ d^2y + (\mathrm{P}'dx + \mathrm{Q}'dy)dt + (\mathrm{R}'x + \mathrm{S}'y)dt^2 &= \mathrm{T}'dt^2 \end{aligned}\right\}.$$

Substitute $dx = pdt$, $dy = qdt$; then we shall have four equations between the five variables x, y, t, p and q; viz.

$$\left.\begin{aligned} dp + (\mathrm{P}p + \mathrm{Q}q + \mathrm{R}x + \mathrm{S}y)dt &= \mathrm{T}dt \\ dq + (\mathrm{P}'p + \mathrm{Q}'q + \mathrm{R}'x + \mathrm{S}'y)dt &= \mathrm{T}'dt \end{aligned}\right\} \text{ and } \left.\begin{aligned} dx - pdt &= 0 \\ dy - qdt &= 0 \end{aligned}\right\};$$

which, being integrated, will give p and q, and consequently x and y in terms of t.

32. *Required to integrate a linear system of any order in which all the coefficients are constant, and the function of the independent variable* $= 0$. (Vid. ch. 6, 20.)

Let the proposed consist of three equations of the second order between the variables x, y, z functions of t.

Substitute $x = e^{at}$, $y = \alpha e^{at}$, $z = \beta e^{at}$; then the result is divisible by e^{at}, and there will be three quadraticks for determining a which contain the constant coefficients and the quantities α and β. Eliminating α and β by means of these quadraticks, there will result an equation of six dimensions, from which six values of a may be obtained. Let them be a', $a'' \cdots a^6$. Also let the corresponding values of α and β, as determined from the same equations, be α', $\alpha'' \cdots \alpha^{6}$ and β', $\beta'' \cdots \beta^{6}$; then it may be shown as in ch. 6, 20, that the primitive is

$$\left.\begin{aligned} x &= c'e^{a't} + c''e^{a''t} + \cdots + c^{6}e^{a^{6}t} \\ y &= c'\alpha'e^{a't} + c''\alpha''e^{a''t} + \cdots + c^{6}\alpha^{6}e^{a^{6}t} \\ z &= c'\beta'e^{a't} + c''\beta''e^{a''t} + \cdots + c^{6}\beta^{6}e^{a^{6}t} \end{aligned}\right\}.$$

The same method is applicable to a system of any order; and the cases of equal and of impossible roots may be provided for as before. For other methods of integration, vid. Garnier's Calc. Int. p. 345.

33. EXAMPLES.

Ex. 1. $$\left.\begin{aligned} dx + (ax + by)dt &= 0 \\ dy + (a'x + b'y)dt &= 0 \end{aligned}\right\}.$$

Here $dx + \vartheta dy + ((a + a'\vartheta)x + (b + b'\vartheta)y)dt = 0$.

Substitute $z = x + \vartheta y$; then, eliminating x, we have

$$dz + (a + a'\vartheta)zdt - y\{d\vartheta + ((a + a'\vartheta)\vartheta - (b + b'\vartheta))dt\}.$$

Assume $d\theta = 0$ and $(a + a'\theta)\theta - (b + b'\theta) = 0$. Let the roots of this quadratick be θ and θ'; then, since $dz + (a + a'\theta)zdt = 0$, we have, integrating,

$z = ce^{-(a+a'\theta)t} = ce^{-a'\theta t}$ (for instead of θ constant and t variable, we are at liberty to suppose θ variable and t constant); and consequently the required primitive consists of

$$\left.\begin{aligned} x + \theta y &= ce^{-a'\theta t} \\ x + \theta' y &= c'e^{-a'\theta' t} \end{aligned}\right\}, \text{ or}$$

$$x = \frac{c'\theta e^{-a'\theta' t} - c\theta' e^{-a'\theta t}}{\theta - \theta'} \text{ and } y = \frac{ce^{-a'\theta t} - c'e^{-a'\theta' t}}{\theta - \theta'}.$$

If $\theta = \theta'$, these values of x and y become infinite and fail; but in this case there is only one value of z, viz. $z = x + \theta y$, which therefore $= ce^{-a'\theta t}$, or $x = ce^{-a'\theta t} - \theta y$. Substitute this in the second proposed equation, and there results $dy + (b' - a'\theta)ydt + ca'e^{-a'\theta t}dt = 0$; wherefore integrating,

$$y = e^{-(b'-a'\theta)t}\left\{ c' - \frac{ca'e^{(b'-2a'\theta)t}}{b'-2a'\theta} \right\} = c'e^{a'\theta t} - \frac{ca'e^{-a'\theta t}}{a}.$$

The value of x in terms of t may be obtained from $x = ce^{-a'\theta t} - \theta y$.

If the roots of the quadratick are impossible, x and y are to be expressed in circular functions of t as in ch. 6, 21.

It appears then that x and y may be each expressed in terms of t and of two constants, which may be determined if any corresponding values of the three variables be known.

Ex. 2. $$\left.\begin{aligned} d^2x + (ax + by + c)dt^2 &= 0 \\ d^2y + (a'x + b'y + c')dt^2 &= 0 \end{aligned}\right\}.$$

We shall integrate this not as in art. 31, but by a method similar to the last.

Multiplying the second equation by θ and adding, there results

$$d^2x + \theta d^2y + \left\{(a + a'\theta)x + (b + b'\theta)y + c + c'\theta\right\}dt^2 = 0 \quad (1).$$

Substitute $z = x + \dfrac{b + b'\theta}{a + a'\theta}y + \dfrac{c + c'\theta}{a + a'\theta}$; then differentiating and assuming $d\theta = 0$,

$$d^2z = d^2x + \frac{b + b'\theta}{a + a\theta}d^2y = d^2x + \theta d^2y + \left\{\frac{b + b'\theta}{a + a\theta} - \theta\right\}d^2y.$$

Substituting this in (1), and assuming

$\frac{b+b'\theta}{a+a'\theta}-\theta=0$ (2), there results $d^2z + (a + a'\theta)zdt^2 = 0$ (3), whose primitive is $z = \text{csin.A}^{\frac{1}{2}}t + \text{c'cos.A}^{\frac{1}{2}}t$, if $\text{A} = a + a'\theta$ (pp. 186, 220).

If it be required to obtain t as a function of (x, y), we have, multiplying (3) by $2dz$, $\frac{2dzd^2z}{dt^2} + 2\text{A}zdz = 0$ ∴ integrating, $\frac{dz^2}{dt^2} + \text{A}z^2 = c^2$, or $dt = \frac{dz}{\sqrt{c^2-\text{A}z^2}}$; which, when integrated, will give the two requisite values of t by finding θ and θ' from equation (2).

Ex. 3. $$\left.\begin{aligned} \frac{2d^2x}{dt^2}+\frac{dx}{dt}+x+\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=0 \\ \frac{d^2x}{dt^2}+96\frac{dx}{dt}-9x+\frac{d^2y}{dt^2}+50\frac{dy}{dt}+15y=0 \end{aligned}\right\}.$$

Substitute $x = e^{at}$; $y = \alpha e^{at}$; and there will result

$$\left.\begin{aligned} 2a^2 + a + 1 + \alpha(a^2 + a + 1) = 0 \\ a^2 + 96a - 9 + \alpha(a^2 + 50a + 15) = 0 \end{aligned}\right\};$$

and eliminating α, we have $\frac{2a^2+a+1}{a^2+a+1} = \frac{a^2+96a-9}{a^2+50a+15}$, or $a^4 + 4a^3 - 7a^2 - 22a + 24 = 0$; whose roots may be found by the method of divisors (Alg. 336) to be 2, 1, −3, −4; and the corresponding values of α are

$-\frac{11}{7}, -\frac{4}{3}, -\frac{16}{7}, -\frac{29}{13}$; wherefore

$$\left.\begin{aligned} x &= c'e^{2t} + c'^2e^{t} + c'^3e^{-3t} + c'^4e^{-4t} \\ -y &= \frac{11c'e^{2t}}{7} + \frac{4c'^2e^{t}}{3} + \frac{16c'^3e^{-3t}}{7} + \frac{29c'^4e^{-4t}}{13} \end{aligned}\right\}.$$

34. MISCELLANEOUS PRAXIS.

1. $\frac{ydx}{a-z} + \frac{xdy}{a-z} + \frac{xydz}{(a-z)^2} = 0 \therefore \frac{xy}{a-z} = c.$
2. $dx+dy + dz + (x + y+z)dz=0 \therefore (x+y+z)e^z=c.$
3. $(ay - bz)dx + (cz - ax)dy + (bx - cy)dz = 0 \therefore$ $\text{c} = \frac{ay-bz}{cz-ax}.$

4. $(2x + z)dx + (2y + z)dy + (x + y)dz = 0 \therefore$ . . . $x(x + z) + y(y + z) = c$.

5. $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0 \therefore$ $c = \dfrac{xy+yz}{y+z}$.

6. $2(y + z)dx + (x + 3y + 2z)dy + (x + y)dz = 0 \therefore$ $(x + y)^2(y + z) = c^3$.

7. $(yz+z^2)dx+(zt-xz)dy+(xy-yt)dz+(y^2+yz)dt=0$ $\therefore xz + yt = c(y + z)$.

8. $du = \dfrac{1}{z^2}(bydz + azdx - bzdy - axdz) \therefore$ . . . $u = \dfrac{ax-by}{z}$.

9. $du = \dfrac{xdy-xdz+zdx-ydx}{(x-y+z)^2} \therefore u = \dfrac{x}{x-y+z} + c$, or $= c - \dfrac{z-y}{x-y+z}$.

10. $du = \dfrac{yz(y+z)}{(x+y+z)^2}dx + \dfrac{xz(x+z)}{(x+y+z)^2}dy + \dfrac{xy(x+y)}{(x+y+z)^2}dz$ $\therefore u = \dfrac{xyz}{x+y+z}$.

11. $ayd^2y + ady^2 + Pdydx + Qdx^2$ is an exact fluxion when $\dfrac{dP}{dx} = \dfrac{dQ}{dy}$.

12. $3ax^3dy^2d^2y - bxdx^2d^2y + 3ax^2dy^3dx - bdydx^3 + cxdydx^3$ $+ cydx^4$ is an exact fluxion; and $u = ax^3p^3 - bxp$ $+ cxy + c$.

13. $y^2d^2x - (x^2 + xy)d^2y - xdy^2 - (2x - y)dxdy = dv$ $\therefore v = y^2dx - (x^2 + xy)dy + c$.

14. $(axy+x^2)d^2x+x^2d^2y+(a+2)xdxdy+(ay+2x)dx^2=dv$ $\therefore v = x^2dx + x^2dy + axydx + c$.

15. $ayd^2x + (ax - 2y)d^2y + 2adxdy - 2dy^2$ is a second exact fluxion; and $u = axy - y^2 + c$.

16. $dz = aydx + bdy \therefore \phi z = ax + bly$, $\phi'z = \dfrac{1}{y}$.

17. $dz = xy(xdx + ydy) \therefore \phi z = x^2 + y^2$, $\phi'z = \dfrac{2}{xy}$.

18. $ydy + ydx + xdz = 0 \therefore$

$$\left.\begin{array}{l} z = \phi(x+y) \\ y + x\phi'(x+y) = 0 \end{array}\right\} \text{ and } \left.\begin{array}{r} x(z+y) + y^2 = y\phi\dfrac{x}{y} \\ z = \phi'\dfrac{x}{y} \end{array}\right\}.$$

19. $dz = 2dx + xdy \therefore \left\{ \phi z + yl.x^2 = 0 \text{ and } x\phi' z + 1 = x \right\}.$

20. $dz = \dfrac{2ydy + xdy + ydx}{a+x+y} \therefore \left.\begin{array}{l} \phi z = y(x+y) \\ \phi' z = a + x + y \end{array}\right\}.$

21. $zdx + xdy + ydz = 0 \therefore y = \mathrm{x}$

$$z = e^{-\int\frac{dx}{x}} \left\{ c - \int e^{\int\frac{dx}{x}} \frac{xd\mathrm{x}}{\mathrm{x}} \right\} \Bigg\}.$$

22. $dyd^2z - dzd^2y = xdx^2dy \therefore z = \int \dfrac{x\phi x dx}{\phi' x}$

$$y = \int \frac{xdx}{\phi' x} \Bigg\}.$$

23. $xdxd^2z - 2dyd^2y + xdx^3 = 0 \therefore \phi x = z - \dfrac{x^2}{2}$

$$y = \int \sqrt{x^2 + x\phi' x - \phi x}\, dx \Bigg\}.$$

24. $\left.\begin{array}{l} dx + (5x+y)dt = e^t dt \\ dy + (3y - x)dt = e^{2t} dt \end{array}\right\} \therefore$

$$\left.\begin{array}{l} x + y = ce^{-4t} + \dfrac{e^t}{5} + \dfrac{e^{2t}}{6} \\ y = (ct + c')e^{-4t} + \dfrac{e^t}{25} + \dfrac{7e^{2t}}{36} \end{array}\right\}.$$

25. $\left.\begin{array}{l} 4dx + 9dy + (44x + 49y)dt = tdt \\ 3dx + 7dy + (34x + 38y)dt = e^t dt \end{array}\right\} \therefore$

$$\left.\begin{array}{l} y + x = ce^{-6t} - \dfrac{5e^t}{7} + \dfrac{2t}{3} - \dfrac{1}{9} \\ y - 4x = c'e^{-t} + 20e^t - 31t + 31 \end{array}\right\}.$$

26. $\left.\begin{array}{l} d^2x - (2x - 5y)dt^2 = 0 \\ d^2y - (x + 2y)dt^2 = 0 \end{array}\right\} \therefore$

$$x = ce^{\sqrt{\frac{5}{2}}t} \sin.\left(\frac{t}{\sqrt{2}} + c'\right) + c''e^{-\sqrt{\frac{5}{2}}t} \sin.\left(\frac{t}{\sqrt{2}} + c'''\right) \Bigg\}$$

$y = \alpha x$. Or $t = \int \dfrac{dz}{(2+\theta)z^2 + c^2}$, where $\theta = \pm\sqrt{-5}$.

27. $\left.\begin{aligned} d^2x - (3x + 4y - 3)dt^2 &= 0 \\ d^2y + (x - 8y + 5)dt^2 &= 0 \end{aligned}\right\} \therefore$

$$\left.\begin{aligned} x - y + \frac{1}{2} &= ce^{2t} + c'e^{-2t} \\ x - y + \frac{17}{7} &= c''e^{t\sqrt{7}} + c'''e^{-t\sqrt{7}} \end{aligned}\right\}.$$

28. $\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{d^2y}{dt^2} &= x + 16y - 3 \\ \frac{d^2y}{dt^2} + 2\frac{d^2x}{dt^2} &= 2y - 3x - 1 \end{aligned}\right\} \therefore$

$$\left.\begin{aligned} 5x + 20y - 3 &= c\sin.t\sqrt{5} + c'\cos.t\sqrt{5} \\ x + y &= c''\sin.t\sqrt{2} + c'''\cos.t\sqrt{2} \end{aligned}\right\}.$$

29. $\left.\begin{aligned} 7d^2x + 5d^2y - (16x + 23y - 46)dt^2 &= 0 \\ 11d^2x + 8d^2y - (25x + 36y - 73)dt^2 &= 0 \end{aligned}\right\} \therefore$

$$\left.\begin{aligned} x + 2y - 13 &= ce^t + c'e^{-t} \\ y &= c''e^t + c'''e^{-t} - \frac{t}{2}(ce^t - c'e^{-t}) + 18t \end{aligned}\right\}.$$

30. $\left\{\frac{d^2x}{dt^2} = 0;\ \frac{d^2y}{dt^2} = 0;\ \frac{d^2z}{dt^2} = 0\right\} \therefore$

$$\{x = a + bt;\ y = a' + b't;\ z = a'' + b''t\}.$$

31. $$\left.\begin{aligned} &\frac{2d^2x}{dt^2} - \frac{18dx}{dt} + 58x + \frac{3d^2y}{dt^2} + \frac{3dy}{dt} \quad . \quad . \quad . \quad . \\ &- 48y + \frac{d^2z}{dt^2} - \frac{3dz}{dt} + 2z = 0 \quad . \quad . \quad . \quad . \quad . \\ &\frac{d^2x}{dt^2} + \frac{43dx}{dt} - 254x - \frac{23d^2y}{dt^2} + \frac{dy}{dt} \quad . \quad . \quad . \\ &+ 250y + \frac{d^2z}{dt^2} + \frac{17dz}{dt} - 36z = 0 \quad . \quad . \quad . \quad . \\ &\frac{3d^2x}{dt^2} + \frac{11dx}{dt} - 128x - \frac{12d^2y}{dt^2} + \frac{2dy}{dt} \quad . \quad . \quad . \\ &+ 130y - \frac{d^2z}{dt^2} + \frac{13dz}{dt} - 18z = 0 \quad . \quad . \quad . \end{aligned}\right\} \therefore$$

$$\left.\begin{aligned} x &= ce^t + c'e^{2t} + c''e^{3t} + c'''e^{4t} + c^{\text{iv}}e^{-t} + c^{\text{v}}e^{-2t} \\ y &= ce^t + c'e^{2t} + 2c''e^{3t} - c'''e^{4t} + 2c^{\text{iv}}e^{-t} + 3c^{\text{v}}e^{-2t} \\ z &= ce^t + 2c'e^{2t} + c''e^{3t} - c'''e^{4t} + 3c^{\text{iv}}e^{-t} + 2c^{\text{v}}e^{-2t} \end{aligned}\right\}.$$

(Paoli, Elementi d'Algebra, tom. 2, p. 192).

CHAPTER VIII.

Partial Fluxional Equations.

1. *Def.* A PARTIAL fluxional equation of three or more variables is one which has arisen from differentiating partially on the hypothesis that one of the variables is a function of the others.

The form of the equation indicates the hypothesis on which the differentiation has been effected; and our only restriction in integrating is that we make not any supposition inconsistent with the hypothesis.

2. Partial equations are attended with difficulties peculiar to themselves; and they are said to be integrated when their integration is made to depend upon that of a total equation.

3. An equation of two variables has only one fluxional coefficient in each order; but an equation of three variables, when differentiated partially, has two fluxional coefficients of the first order, three of the second, four of the third, and so on. Hence if $z=f(x,y)$, the canonical form of a partial fluxional equation of the first order is

$$\text{F}\left(\frac{dz}{dx}, \frac{dz}{dy}, x, y, z\right) = 0.$$

4. *Required to integrate* $\frac{dz}{dx} = \text{R}$, *where* $\text{R} = \text{F}(x, y, z)$.

It appears from the form of the proposed that it has arisen from differentiating $z = f(x, y)$ partially with respect to x; wherefore in integrating, the arbitrary constant is to be a function of y.

Now $\frac{dz}{dx} = \text{R}$ or $(dz) = \text{R}dx$, if dz be enclosed in brackets to denote its value when y is constant; and since this is an equation between two variables z and x, it is to be inte-

grated as in ch. 4, and a correction is to be added which is an arbitrary function of y.

The function of y in this case is as arbitrary as possible; for it is not under any restriction whatever.

Cor. If more than three variables enter into the function R, the correction is an arbitrary function of all the variables except z and x.

Ex. 1. $\frac{dz}{dx} = a.$

If this be a partial equation, we have $(dz) = adx$; and integrating, $z = ax + \phi y$; where it is not necessary to add the constant, since it is contained in ϕy.

Ex. 2. $\frac{dz}{dx} = x^2 + a^2 \therefore z = \frac{x^3}{3} + a^2x + \phi y.$

Ex. 3. $\frac{dz}{dx} = \text{Y}$, where $\text{Y} = fy.$

This is necessarily a partial equation; and we have $(dz) = \text{Y}dx$; and integrating, $z = \text{Y}x + \phi y$.

Ex. 4. $\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} \therefore z = \sqrt{x^2 + y^2} + \phi y.$

Ex. 5. $\frac{dz}{dx} = \frac{x^2-y^2}{x^2+y^2} \cdot \frac{z}{x}.$

Here $\frac{(dz)}{z} = \frac{x^2-y^2}{x^2+y^2}\frac{dx}{x} = \frac{2x^2-(x^2+y^2)}{x^2+y^2}\frac{dx}{x} = \frac{2xdx}{x^2+y^2} - \frac{dx}{x};$

and integrating, $z = \frac{x^2+y^2}{x}\phi y.$

Ex. 6. $\frac{dz^2}{dx^2} = x + y + t;$ where $z = \text{F}(x, y, t).$

Here $(dz) = dx\sqrt{x+y+t} \therefore z = \frac{2}{3}(x+y+t)^{\frac{3}{2}} + \phi(y, t).$

5. *Required to integrate* $\frac{dz}{dy} = \text{R}$, *where* $\text{R} = \text{F}(x, y, z).$

Here $(dz) = \text{R}dy$ is to be integrated as if it were an equation between two variables z and y; and a correction is to be added which is an arbitrary function of x.

6. *Required to integrate* $\text{P}\frac{dz}{dx} + \text{Q}\frac{dz}{dy} = \text{R}$, *where* P, Q *and* R *are each functions of* (x, y).

Substituting $p = \frac{dz}{dx}$, $q = \frac{dz}{dy}$, the proposed is $\mathrm{P}p + \mathrm{Q}q = \mathrm{R}$.

It appears from the form of the equation that $z = \mathrm{F}(x, y)$; and consequently $dz = pdx + qdy =$, eliminating p,

$$\frac{\mathrm{R} - \mathrm{Q}q}{\mathrm{P}}dx + qdy = \frac{\mathrm{R}dx}{\mathrm{P}} - \frac{q}{\mathrm{P}}(\mathrm{Q}dx - \mathrm{P}dy).$$

Now $\mathrm{Q}dx - \mathrm{P}dy$ being a function which contains only two variables, let t be the factor which renders it an exact fluxion $= ds$; then $dz = \frac{\mathrm{R}dx}{\mathrm{P}} - \frac{q}{t\mathrm{P}}ds$.

Since $s = f(x, y)$, we may by means of it eliminate y from P, Q and R. Let them by this process become P', Q' and R'; then we have $dz = \frac{\mathrm{R}'dx}{\mathrm{P}'} - \frac{q}{t\mathrm{P}'}ds$; and integrating on the hypothesis that s is constant, $z = \int \frac{\mathrm{R}'dx}{\mathrm{P}'} + \varphi s$; which is therefore the required primitive.

If we eliminate q, we shall have

$$dz = pdx + \frac{\mathrm{R} - \mathrm{P}p}{\mathrm{Q}}dy = \frac{\mathrm{R}dy}{\mathrm{Q}} + \frac{p}{\mathrm{Q}}(\mathrm{Q}dx - \mathrm{P}dy) = \frac{\mathrm{R}dy}{\mathrm{Q}} + \frac{p}{t\mathrm{Q}}ds.$$

Let P, Q and R become P', Q' and R' when x is eliminated by means of $s = f(x, y)$; then, integrating on the hypothesis that s is constant, $z = \int \frac{\mathrm{R}'dy}{\mathrm{Q}'} + \varphi s$; which is sometimes a more convenient form than the preceding.

Cor. The primitive of $\mathrm{P}\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} = 0$ is $z = \varphi s$.

This may be verified by reversing the process; for, since $z = \phi s$, $\frac{dz}{dx} = \varphi' s \frac{ds}{dx} = \varphi' s . t\mathrm{Q}$. Similarly $\frac{dz}{dy} = -\varphi' s \, . \, t\mathrm{P}$; and eliminating $\varphi' s$, there results $\mathrm{P}\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} = 0$.

Ex. 1. $\frac{dz}{dx} = \frac{dz}{dy}$.

Here $p = q \therefore dz$, which $= pdx + qdy$, $= pd(x + y) \therefore p = \varphi'(x + y)$ and $z = \varphi(x + y)$. (Vid. p. 110, note.)

Ex. 2. $x\frac{dz}{dx} + y\frac{dz}{dy} = 0.$

This equation belongs to all conical surfaces whatever, the origin of the co-ordinates being at the vertex of the cone.

Here $Qdx - Pdy$ is $ydx - xdy \therefore t = \frac{1}{y^2}$ and $s = \frac{x}{y} \therefore$ $z = \phi\frac{x}{y}$.

Ex. 3. $y\frac{dz}{dx} - x\frac{dz}{dy} = 0.$

This is the equation of all surfaces of revolution, the axis of revolution coinciding with z.

Here $Qdx - Pdy$ is $xdx + ydy \therefore s = x^2 + y^2$ and $z = \phi(x^2 + y^2)$. This may be verified by differentiating z partially with respect both to x and y, and eliminating $\phi'(x^2 + y^2)$.

Ex. 4. $\frac{dz}{dx} + \frac{x^2+y^2}{2xy}\frac{dz}{dy} = 0.$

Here $Qdx - Pdy$ is $\frac{x^2+y^2}{2xy}dx - dy$; and to find the factor which will render this an exact fluxion, the variables must be first separated (ch. 4, 22, *Cor.* 2); but since it is homogeneous, this may be effected by substituting $y = xv$, when it becomes $\frac{1+v^2}{2v}dx - (vdx + xdv)$ which

$= \frac{dx}{2v} - \frac{vdx}{2} - xdv = -\frac{1}{2v}(v^2dx + 2xvdv - dx) = \ \cdot\ \cdot\ \cdot$

$-\frac{1}{2v}d(v^2x - x)$; or the requisite factor is $-2v = -\frac{2y}{x}$; and multiplying by this, $Qdx - Pdy$ becomes

$\frac{2ydy}{x} - \frac{y^2dx}{x^2} - dx$, which $= d.\frac{y^2-x^2}{x}$; wherefore

$z = \phi\frac{y^2-x^2}{x}$. (Vid. vol. i. ch. 4. 39. Praxis, ex. 1.)

Ex. 5. $\frac{dz}{dx} + a\frac{dz}{dy} = b.$

Here $Qdx - Pdy$ is $adx - dy \therefore s = y - ax \therefore \ \cdot\ \cdot\ \cdot$ $z - bx = \phi(y - ax)$.

Ex. 6. $xy\frac{dz}{dy} - x^2\frac{dz}{dx} = y^2$.

Here $\mathrm{Q}dx - \mathrm{P}dy$ is $xydx + x^2dy \therefore s = xy \therefore y = \frac{s}{x} \therefore \frac{\mathrm{R}}{\mathrm{P}}$, which $= -\frac{y^2}{x^2}, = -\frac{s^2}{x^4} \therefore z = \frac{s^2}{3x^3} + \phi s = \frac{y^2}{3x} + \phi.xy$.

Ex. 7. $\frac{dz}{dx} + y\frac{dz}{dy} = x + y$.

Here $\mathrm{Q}dx - \mathrm{P}dy$ is $ydx - dy \therefore s = ly - x$, or $y = e^{s+x}$; and the primitive is

$$z = \int(x + e^{s+x})dx + \phi s = \frac{x^2}{2} + e^{s+x} + \phi(ly - x) \cdot \cdot \cdot$$

$$= \frac{x^2}{2} + y + \psi.ye^{-x};$$ for, since $l.ye^{-x} = ly - x$, therefore $\phi(ly - x) = \phi l.ye^{-x} = \psi.ye^{-x}$.

Ex. 8. $\frac{dz}{dx} + \mathrm{X}\frac{dz}{dy} = \mathrm{R}$, where $\mathrm{R} = \mathrm{F}(x, y)$.

Here $s = \int \mathrm{X}dx - y$, or $y = \int \mathrm{X}dx - s \therefore \mathrm{R}' = \mathrm{F}(x, \int \mathrm{X}dx - s) \therefore z = \int \mathrm{F}(x, \int \mathrm{X}dx - s)dx + \phi s$, where $s = \int \mathrm{X}dx - y$.

7. *Required to integrate* $\mathrm{P}\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} = \mathrm{R}$, *where* P *and* Q *are functions of* (x, y) *and* R *is a function of* (x, y, z).

Eliminating p by means of $dz = pdx + qdy$, we have

$$dz = \frac{\mathrm{R}dx}{\mathrm{P}} - \frac{q}{\mathrm{P}}(\mathrm{Q}dx - \mathrm{P}dy), \text{ or}$$

$$q(\mathrm{Q}dx - \mathrm{P}dy) - (\mathrm{R}dx - \mathrm{P}dz) = 0.$$

Let t as before be the factor which renders $\mathrm{Q}dx - \mathrm{P}dy$ an exact fluxion $= ds$, and t' the factor which renders $\mathrm{R}dx - \mathrm{P}dz$, when y has been eliminated from R, an exact fluxion $= ds'$; then

$$\frac{q}{t}ds - \frac{1}{t'}ds' = 0, \text{ or } ds = \frac{t}{qt'}ds'; \text{ which cannot be unless}$$

$$\frac{t}{qt'} = \phi's' \text{ and } s = \phi s'; \text{ which is therefore the required primitive.}$$

If we eliminate q, there will result

$$p(\mathrm{Q}dx - \mathrm{P}dy) + \mathrm{R}dy - \mathrm{Q}dz = 0.$$

Let t' be the factor which renders $\mathrm{R}dy - \mathrm{Q}dz$, when x has

been eliminated from R, an exact fluxion $= ds'$; then it may be shown as before that the primitive is $s = \phi s'$.

Hence it appears that all equations of the above form are integrable, *i. e.* reducible to the integration of equations of the first order between two variables.

We shall subjoin another mode of integrating the same form by means of an indeterminate function.

8. *Required to integrate* $\mathrm{P}\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} = \mathrm{R}$, *where* P *and* Q *are functions of* (x, y), *and* R *is a function of* (x, y, z).

Assume θ such that $z = f(x, \theta)$, where θ is a function of (x, y) hereafter to be determined.

Let $\left(\frac{dz}{dx}\right)$ represent a fluxional coefficient of $z = f(x, \theta)$, which is enclosed in brackets to distinguish it from the fluxional coefficient of $z = \mathrm{F}(x, y)$; then, differentiating $z = f(x, \theta)$ partially, we have $\frac{dz}{dx} = \left(\frac{dz}{dx}\right) + \frac{dz}{d\theta}\frac{d\theta}{dx}$ and $\frac{dz}{dy} = \frac{dz}{d\theta}\frac{d\theta}{dy}$; wherefore by substitution the proposed becomes

$$\mathrm{P}\left(\frac{dz}{dx}\right) + \frac{dz}{d\theta}\left\{\mathrm{P}\frac{d\theta}{dx} + \mathrm{Q}\frac{d\theta}{dy}\right\} = \mathrm{R}.$$

Assume θ such that $\mathrm{P}\frac{d\theta}{dx} + \mathrm{Q}\frac{d\theta}{dy} = 0$ (1); then $\mathrm{P}\left(\frac{dz}{dx}\right) = \mathrm{R}$ (2). The equation (1) can be integrated as in art. 6; from which we obtain $\theta = \phi(x, y)$ an arbitrary function of (x, y) of a known form. Also in integrating (2), since $\left(\frac{dz}{dx}\right)$ is the fluxional coefficient which has arisen from supposing θ to be the only constant, the correction must be $\chi\theta$ or $\chi\phi(x, y) = \psi(x, y)$ where $\psi(x, y)$ represents an arbitrary function of a known form of (x, y).

Ex. $\frac{dz}{dx} + \frac{y}{x}\frac{dz}{dy} = x^m z^n$.

Here equation (1) is $\frac{d\theta}{dx} + \frac{y}{x}\frac{d\theta}{dy} = 0 \therefore$ (6, ex. 2)

$\theta = \phi\frac{y}{x}$. Also $(dz) = x^m z^n dx$, or $z^{-n}\, dz = x^m dx$; and integrating, $\frac{1}{(n-1)z^{n-1}} = \psi\frac{y}{x} - \frac{x^{m+1}}{m+1}$.

9. *Required to integrate* $\mathrm{P}\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} = \mathrm{R}$, *where* P, Q *and* R *are each functions of* (x, y, z).

Eliminating both p and q by means of $dz = pdx + ydy$, we have

$q(\mathrm{Q}dx - \mathrm{P}dy) - (\mathrm{R}dx - \mathrm{P}dz) = 0$ (1)
$p(\mathrm{Q}dx - \mathrm{P}dy) + \mathrm{R}dy - \mathrm{Q}dz = 0$ (2) ; and by combining these,

$p(\mathrm{R}dx - \mathrm{P}dz) - q(\mathrm{R}dy - \mathrm{Q}dz) = 0$ (3).

Suppose that any of the three functions $\mathrm{Q}dx - \mathrm{P}dy$ (α), $\mathrm{R}dx - \mathrm{P}dz$ (β), $\mathrm{R}dy - \mathrm{Q}dz$ (γ) contains only two of the variables. Let it be (γ); then if t be the factor which renders it an exact fluxion $= ds$, we have from equation (3),

$$p(\mathrm{R}dx - \mathrm{P}dz) - \frac{q}{t}ds = 0.$$

Let R and P become R′ and P′ when y is eliminated by means of s which is a function of (y, z), and let t' be the factor which renders $\mathrm{R}'dx - \mathrm{P}'dz$ an exact fluxion $= ds'$; then $\frac{p}{t'}ds' - \frac{q}{t}ds = 0$, or $ds = \frac{pt}{qt'}ds'$, and consequently the required primitive is $s = \phi s'$.

It appears then that the proposed is always integrable if either of the three functions (α), (β) or (γ) contains only two of the variables.

The case in which each of the three equations contains the three variables shall be integrated in art. 16.

Cor. If P = 0; the equation $\mathrm{R}dx - \mathrm{P}dz = 0$ gives $dz = 0$, or $z = a$. Substitute this value of z in $\mathrm{Q}dx - \mathrm{P}dy$, which then becomes an equation between two variables. Let it be rendered an exact fluxion $= ds$; then, the required primitive is $z = \phi s$, where s results from the integration of $\mathrm{Q}dx - \mathrm{P}dy = 0$, in which z is considered constant.

10. It is observable that the equations which result from assuming y and x to be separately constant in the form of the preceding article are $\mathrm{P}\frac{dz}{dx} = \mathrm{R}$, or $\mathrm{R}dx - \mathrm{P}dz = 0$ and

$\text{Q}\dfrac{dz}{dy} = \text{R}$, or $\text{R}dy - \text{Q}dz$; and by combining these, $\text{Q}dx - \text{P}dy = 0$; which are severally the same as (α), (β) and (γ). Either of these equations may be assumed; and if it contain only two of the variables, the integration may be effected as has been shown; but we may not assume any two of them to obtain simultaneously and then integrate, for the hypotheses of integration would be inconsistent. In (γ), x is constant, but in (β) it is variable. Also in (α) both x and y are supposed to be variable.

Cor. 1. Since we may assume either $ds = 0$ or $ds' = 0$, we have $s = a$ or $s' = b$, where a and b are the arbitrary constants of two of the equations (α), (β) and (γ).

Cor. 2. Conversely, if a and b are the arbitrary constants of two of these equations, the primitive is $a = \phi b$ where a and b are each functions of two of the variables.

11. EXAMPLES.

Ex. 1. $x\dfrac{dz}{dx} + y\dfrac{dz}{dy} = nz.$

Here (α) is $ydx - xdy \therefore s = \dfrac{y}{x}$. Also (β) is $nzdx - xdz \therefore s' = l\dfrac{z}{x^n}$, and the primitive is $\dfrac{z}{x^n} = \phi\dfrac{y}{x}$, or $z = x^n\phi\dfrac{y}{x}$ = an homogeneous function of (x, y) of n dimensions (ch. 4, 6).

Ex. 2. $az\dfrac{dz}{dx} - xz\dfrac{dz}{dy} + xy = 0.$

Here (α) is $xzdx + azdy = 0$; (β) is $xydx + azdz = 0$ and (γ) is $xydy - xzdz = 0$; $\therefore$ the primitive is $z^2 - y^2 = \phi(x^2 + 2ay)$.

Ex. 3. $xy\dfrac{dz}{dx} + y^2\dfrac{dz}{dy} + xz = axy\sqrt{x^2 + y^2}.$

Here (α) is $y^2dx - xydy = 0$, or $s = \dfrac{y}{x}$. Also (β) is $(axy\sqrt{x^2 + y^2} - xz)dx - xydz = 0$, or $dz + \dfrac{zdx}{y} = a\sqrt{x^2 + y^2}dx$, and eliminating y, ...

$dz + \frac{zdx}{sx} = a\sqrt{1+s^2}xdx$; whence

$$z = x^{-\frac{1}{s}}\left\{\phi s + a\sqrt{1+s^2}\int x^{\frac{1}{s}+1}dx\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$= \frac{axy\sqrt{x^2+y^2}}{x+2y} + x^{-\frac{x}{y}}\phi\frac{y}{x}.$$

12. *All homogeneous equations of the first order and degree are integrable.*

Let the equation be reduced to the form $\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} = \mathrm{R}$.

Assume $x = vz$, $y = wz$; then, by substitution, it becomes $p + \mathrm{v}q = \mathrm{w}$, where v and w do not contain z. Also the equations (β) and (γ) of art. 9 become

$$\left.\begin{array}{l}\mathrm{w}d.vz - dz = 0\\ \mathrm{w}d.wz - \mathrm{v}dz = 0\end{array}\right\} \text{or} \left.\begin{array}{l}(v\mathrm{w}-1)dz + zwdv = 0\ (\beta)\\ (w\mathrm{w}-\mathrm{v})dz + zwdw = 0\ (\gamma)\end{array}\right\};$$

and eliminating dz, there results

$(w\mathrm{w} - \mathrm{v})dv - (v\mathrm{w} - 1)dw = 0$ (α), an equation which contains only two variables.

Let (α), when rendered an exact fluxion, be $= ds$; then, eliminating either v or w from (β) or (γ), there will result an equation which contains only two variables and s. Let this be integrated on the hypothesis that s is constant; and replacing v and w by their values $v = \frac{x}{z}$, $w = \frac{y}{z}$, there will result the required primitive.

13. *Required to integrate the linear equation* . . .

$$\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} + \mathrm{R}z = \mathrm{S}.$$

Here $p + \mathrm{Q}q = -\mathrm{R}z + \mathrm{S}$; and (α) and (β) are $\mathrm{Q}dx - dy = 0$ and $(\mathrm{S} - \mathrm{R}z)dx - dz = 0$.

Let (α) which contains only two variables, when rendered an exact fluxion, be $= ds$; and suppose that S and R, when y is eliminated by means of $s = f(x, y)$, become S′ and R′; then (β) is $dz + \mathrm{R}'zdx = \mathrm{S}'dx$ a linear equation; and consequently the required primitive is

$z = e^{-\int \mathrm{R}'dx}\left\{\int e^{\mathrm{R}'dx}\mathrm{S}'dx + \phi s\right\}$.

Cor. The primitive of $\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} + \mathrm{R}z = 0$ is $z = e^{-\int \mathrm{R}'dx}\phi s$.

Ex. 1. $y\frac{dz}{dx} + x\frac{dz}{dy} = nz.$

Here $yp + xq = nz$ $\therefore$ (α) is $xdx - ydy = 0$, or

$s = y^2 - x^2$ $\therefore$ (β) is $dz - \frac{n}{\sqrt{s+x^2}}zdx = 0$, or

$\frac{dz}{z} = \frac{ndx}{\sqrt{s+x^2}}$ $\therefore z = (x + \sqrt{s + x^2})^n \phi s = (x+y)^n \phi(y^2 - x^2).$

Ex. 2. $(ax + by)\frac{dz}{dx} + (cx + gy)\frac{dz}{dy} = kz.$

Here (α) is $(cx + gy)dx - (ax + by)dy = 0$. Substitute $y = tx$; then (α) becomes
$(c + gt)dx - (a + bt)(tdx + xdt) = 0$, or

$\frac{2dx}{x} + \frac{2(a+bt)dt}{bt^2+(a-g)t-c} = 0 = ds.$

To integrate this, let $lu = \int \frac{dt}{bt^2+(a-g)t-c}$, or

$u = \left(\frac{t-m}{t-n}\right)^{\frac{1}{b(m-n)}}$, where m and n are the roots of

$bt^2 + (a - g)t - c = 0$; then

$ds = \frac{2dx}{x} + \frac{2btdt+(a-g)dt}{bt^2+(a-g)t-c} + (a + g)\frac{du}{u}$ $\therefore$ integrating,

$e^s = x^2(bt^2 + (a - g)t - c)u^{a+g}.$

Also (β) is $kzdx - (ax + by)dz = 0$, or

$\frac{kdx}{x} - (a + bt)\frac{dz}{z} = 0$ $\therefore$ $\frac{dz}{z} + \frac{kdu}{u} = 0 = ds'$; and integrating, $zu^k = s'$; and the required primitive is

$z = u^{-k}\phi(x^2(bt^2 + (a - g)t - c)u^{a+g})$ . . . . . .

$= u^{-k}\phi(u^{a+g}(by^2 + (a - g)xy - cx^2))$, where

$u = \left(\frac{y-mx}{y-nx}\right)^{\frac{1}{b(m-n)}}.$

14. *Required to integrate* $P\frac{dz}{dx} + Q\frac{dz}{dy} + R\frac{dz}{dt} = S$, *where* P, Q, R *and* S *are functions of* (x, y, t, z).

The equation becomes by substitution $Pp + Qq + Rr = S$.

Since $z = F(x, y, t)$, $dz = pdx + qdy + rdt$. Eliminating p by means of the proposed, there will result

$q(\mathrm{Q}dx - \mathrm{P}dy) + r(\mathrm{R}dx - \mathrm{P}dt) - (\mathrm{s}dx - \mathrm{P}dz) = 0.$

In this equation q and r are independent quantities, and consequently (Alg. 347), we have $\mathrm{Q}dx - \mathrm{P}dy = 0$ (α), $\mathrm{R}dx - \mathrm{P}dt = 0$ (β) and $(\mathrm{s}dx - \mathrm{P}dz) = 0$ (γ).

First, let (α), (β), (γ), each contain only two variables, and be rendered exact fluxions $= ds$, ds', ds'' by the factors k, k', k'' respectively; then $\frac{q}{k}ds + \frac{r}{k'}ds' - \frac{1}{k''}ds'' = 0$, or

$$ds = -\frac{kr}{qk'}ds' + \frac{k}{qk''}ds''.$$

Since q and r are not only independent but also indeterminate quantities, they might be so assumed that

$-\frac{kr}{qk'} = \phi's'$ and $\frac{k}{qk''} = \chi's''$, in which case

$ds = \phi's'ds' + \chi's''ds''$; and integrating, $s = \phi s' + \chi s''$. But we can also obtain s as a single function of (s', s'') by assuming $-\frac{kr}{qk} = \frac{ds}{ds'}$ and $\frac{k}{qk''} = \frac{ds}{ds''}$, in which case

$ds = \frac{ds}{ds'}ds' + \frac{ds}{ds''}ds''$; and integrating, $s = \phi(s', s'')$.

This result, when under an implicit form, is $\psi(s, s', s'') = 0$; *i. e.* any equation whatever between the quantities s, s' and s''.

By eliminating either q or r, additional equations will arise for determining the primitive. The number of the equations altogether will be six; for there are six combinations of four quantities taken two and two together (Alg. 230); and the proposed is integrable when P, Q, R, S are such functions that any three of the six equations, combined in any manner whatever, are or may be rendered exact fluxions.

Next let only one of the equations as (α) contain only two variables, and by elimination let both (β) and (γ) be expressible in terms of two variables; then as before the primitive is $s = \phi(s', s'')$.

The same method is manifestly applicable where z is a function of any number of variables and the equation is of the first order and degree.

15. EXAMPLES.

Ex. 1. $\mathrm{P}\dfrac{dz}{dx} + \mathrm{Q}\dfrac{dz}{dy} + \mathrm{R}\dfrac{dz}{dt} = 0.$

Eliminating successively p, q and r by means of $dz = pdx + qdy + rdr$, we have

$$-dz = q\left(\frac{\mathrm{Q}}{\mathrm{P}}dx - dy\right) + r\left(\frac{\mathrm{R}}{\mathrm{P}}dx - dt\right)$$

$$-dz = p\left(\frac{\mathrm{P}}{\mathrm{Q}}dy - dx\right) + r\left(\frac{\mathrm{R}}{\mathrm{Q}}dy - dt\right)$$

$$-dz = p\left(\frac{\mathrm{P}}{\mathrm{R}}dt - dx\right) + q\left(\frac{\mathrm{Q}}{\mathrm{R}}dt - dy\right).$$

It appears then that the example is integrable in the three following cases: if the fraction $\dfrac{\mathrm{Q}}{\mathrm{P}}$ contain only x and y, and $\dfrac{\mathrm{R}}{\mathrm{P}}$ only x and t; or if $\dfrac{\mathrm{P}}{\mathrm{Q}}$ contain only x and y, and $\dfrac{\mathrm{R}}{\mathrm{Q}}$ only y and t; or if $\dfrac{\mathrm{P}}{\mathrm{R}}$ contain only x and t, and $\dfrac{\mathrm{Q}}{\mathrm{R}}$ only y and t.

Ex. 2. $\dfrac{dz}{dx} + a\dfrac{dz}{dy} + b\dfrac{dz}{dt} = c.$

The three equations are $adx - dy = 0 = ds$; $bdx - dt = 0 = ds'$ and $cdx - dz = 0 = ds''$, $\therefore$ the primitive is $z - cx = \phi(y - ax, t - bx)$.

To verify this, assume $z = cx + v$; where $v = \phi(p, q)$; $p = y - ax$, $q = t - bx$; then

$$\frac{dz}{dx} = c + \frac{dv}{dx} = c + \frac{dv}{dp}\frac{dp}{dx} + \frac{dv}{dq}\frac{dq}{dx}. \quad \text{(Vol. i. ch. 3, 10.)}$$

$$= c - a\frac{dv}{dp} - b\frac{dv}{dq}$$

$$\frac{dz}{dy} = \frac{dv}{dy} = \frac{dv}{dp}\frac{dp}{dy} = \frac{dv}{dp} \text{ and}$$

$$\frac{dz}{dt} = \frac{dv}{dt} = \frac{dv}{dq}$$

$$\therefore \frac{dz}{dx} + a\frac{dz}{dy} + b\frac{dz}{dt} = c.$$

Ex. 3. $x\frac{dz}{dx} + y\frac{dz}{dy} + t\frac{dz}{dt} = nz.$

Here $\frac{y}{x} = s, \frac{t}{x} = s'$ and $\frac{z}{x^n} = s''$; and the primitive is $z = x^n\phi\left(\frac{y}{x}, \frac{t}{x}\right)$, an homogeneous function of n dimensions.

Ex. 4. $(y + t)\frac{dz}{dx} + (x + t)\frac{dz}{dy} + (x + y)\frac{dz}{dt} = 0.$

The equation $sdx - rdz = 0$, since $s = 0$, gives $dz = 0 = ds \therefore z = s$.

Also two of the equations are

$$\left.\begin{aligned}(x + y)dx - (y + t)dt = 0\\(x + y)dy - (x + t)dt = 0\end{aligned}\right\} \text{ or}$$

$dx = \frac{y+t}{x+y}dt$ and $dy = \frac{x+t}{x+y}dt \therefore dx - dy = -\frac{x-y}{x+y}dt$, or

$$\frac{dx-dy}{x-y} = -\frac{dt}{x+y}.$$

Also $dx + dy = \frac{x+y+2t}{x+y}dt \therefore$ . . . . . .

$$dx + dy + dt = \frac{2(x+y+t)dt}{x+y}, \text{ or}$$

$$\frac{dx+dy+dt}{x+y+t} = \frac{2dt}{x+y} = -\frac{2(dx-dy)}{x-y}; \therefore \text{ integrating,}$$

$(x - y)^2(x + y + t) = s'$.

Similarly $(x - t)^2(x + y + t) = s''$; and the primitive is $z = \phi\{(x - y)^2(x + y + t), (x - t)^2(x + y + t)\}$.

Ex. 5. $x\frac{dz}{dx} + (t + z)\frac{dz}{dy} + (y + z)\frac{dz}{dt} = y + t.$

Three of the equations are

$$\left.\begin{aligned}(t + z)dx - xdy = 0\\(y + z)dx - xdt = 0\\(y + t)dx - xdz = 0\end{aligned}\right\}; \text{ and}$$

adding them together, $2(z+y+t)dx - x(dz+dy+dt) = 0 \therefore$ $\frac{z+y+t}{x^2} = s.$

Also subtracting the first and the third from the second, we have

$(y-t)dx + x(dy-dt) = 0$ } $\therefore xy - xt = s'$ } $\therefore$ the primitive is $z + y + t = x^2\phi(xy - xt, xz - xt)$.
$(z-t)dx + x(dz-dt) = 0$ } $\therefore xz - xt = s''$ }

This result, when not under a resolved form, is

$$\psi\left\{\frac{z+y+t}{x^2}, xy - xt, xz - xt\right\} = 0.$$

To verify the first result, assume $z = -(y+t) + x^2v$, where $v = \phi(p, q)$, $p = xy - xt$, $q = xz - xt$; then

$$\frac{dz}{dx} = 2xv + x^2\left\{\frac{dv}{dp}\frac{dp}{dx} + \frac{dv}{dq}\frac{dq}{dx}\right\}$$

$$= \frac{2(z+y+t)}{x} + x^2\left\{(y-t)\frac{dv}{dp} + \left(z-t+x\frac{dz}{dx}\right)\frac{dv}{dq}\right\}$$

$$\frac{dz}{dy} = -1 + x^2\left\{x\frac{dv}{dp} + x\frac{dz}{dy}\frac{dv}{dq}\right\} \text{ and}$$

$$\frac{dz}{dt} = -1 + x^2\left\{-x\frac{dv}{dp} + \left(x\frac{dz}{dt} - x\right)\frac{dv}{dq}\right\}$$

$$\therefore x\frac{dz}{dx} + (t+z)\frac{dz}{dy} + (y+z)\frac{dz}{dt} - (y+t) \quad . \quad . \quad . \quad .$$

$$= \frac{x^3dv}{dq}\left\{x\frac{dz}{dx} + (t+z)\frac{dz}{dy} + (y+z)\frac{dz}{dt} - (y+t)\right\}, \text{ which}$$

is satisfied by $x\frac{dz}{dx} + (t+z)\frac{dz}{dy} + (y+z)\frac{dz}{dt} = y + t$,

whatever be the form of ϕ.

Ex. 6. $(y+t+z)\frac{dz}{dx} + (x+t+z)\frac{dz}{dy} + (x+y+z)\frac{dz}{dt} \quad . \quad . \quad .$

$= x + y + t.$

Three of the equations are

$(x+y+z)dx - (y+t+z)dt = 0$
$(x+y+z)dy - (x+t+z)dt = 0$ }, or
$(x+y+z)dz - (x+y+t)dt = 0$

$$dx = \frac{y+t+z}{x+y+z}dt;\ dy = \frac{x+t+z}{x+y+z}dt \text{ and } dz = \frac{x+y+t}{x+y+z}dt \therefore$$

$$dx - dt = \frac{t-x}{x+y+z}dt;\ dy - dt = \frac{t-y}{x+y+z}dt \text{ and}$$

$$dz - dt = \frac{t-z}{x+y+z}dt.$$

Also $dx + dy + dz = \frac{3t+2(x+y+z)}{x+y+z}dt$ ∴ . . . . .

$dx + dy + dz + dt = \frac{3(x+y+z+t)}{x+y+z}dt.$

Substitute $v = x + y + z + t$; then $dv = \frac{3v}{x+y+z}dt$, or $\frac{dv}{3v} = \frac{dt}{x+y+z}$ ∴ $\frac{dx-dt}{t-x}$, which $= \frac{dt}{x+y+z}$, $= \frac{dv}{3v}$ or $\frac{dv}{v} + \frac{3(dx-dt)}{x-t} = 0$ ∴ integrating, $v(x-t)^3 = s$. Similarly $v(y-t)^3 = s'$ and $v(z-t)^3 = s''$; and the primitive is $v(z-t)^3 = \phi\{v(x-t)^3, v(y-t)^3\}$, which may be put under the form $\psi\{v(x-t)^3, v(y-t)^3, v(z-t)^3\} = 0$.

16. *The primitive of any partial equation of the first order and degree where one variable is a function of two others is* $a = \phi b$; *where* ϕ *denotes an arbitrary function, and* a *and* b *are the constants of any two of the equations* (α), (β), (γ) *of art.* 9, *considered as simultaneous.*

For let the proposed be, as before, $Pp + Qq = R$. Only two of the equations (α), (β), (γ) are independent: integrating these simultaneously (ch. 7. 28), let their primitives be $\left.\begin{matrix} s = a \\ s' = b \end{matrix}\right\}$, where s and s' are functions of (x, y, z); and a and b are the constants introduced by the integration.

The following demonstration consists in showing that $a = \phi b$ shall satisfy the proposed independently of the form of ϕ.

Assume $a = \phi b$ or $s = \phi s'$; then, since this is an equation between *three* variables, we may (ex hyp.) obtain by differentiation,

$\frac{ds}{dy}dy + \frac{ds}{dz}dz = \phi' s' \left\{\frac{ds'}{dy}dy + \frac{ds'}{dz}dz\right\}$ (vol. i. ch. 4, 32), or

$\frac{ds}{dy} + \frac{ds}{dz}q = \phi' s' \left\{\frac{ds'}{dy} + \frac{ds'}{dz}q\right\}$; and consequently

$q = \frac{\phi' s' \frac{ds'}{dy} - \frac{ds}{dy}}{\frac{ds}{dz} - \phi' s' \frac{ds'}{dz}} =$, by substitution m; which therefore contains the arbitrary function ϕ.

Eliminating p from the proposed equation by means of $dz = pdx + qdy$, we have, as in art. 7,
$\mathrm{R}dx - \mathrm{P}dz - q(\mathrm{Q}dx - \mathrm{P}dy) = 0$, or $\mathrm{P}dz = \mathrm{R}dx - m(\mathrm{Q}dx - \mathrm{P}dy)$.

Now this satisfies the proposed independently of m; for

$$\left.\begin{aligned} \text{we obtain from it, } p, \text{ which } &= \frac{dz}{dx}, \ = \frac{\mathrm{R}}{\mathrm{P}} - \frac{m\mathrm{Q}}{\mathrm{P}} \\ \text{and } q, \text{ which } &= \frac{dz}{dy}, \ = m \end{aligned}\right\} \text{; and by}$$

substitution, the proposed becomes $\mathrm{R} - m\mathrm{Q} + m\mathrm{Q} = \mathrm{R}$; which obtains independently of m or the form of ϕ.

It may be remarked that we arrive at the same conclusion, if we begin by differentiating $s = \phi' s'$ either on the hypothesis that x and z are the only variables, or that x, y, z all vary.

17. *The preceding theorem may be extended to equations containing any number of variables.*

Let the proposed equation be $\mathrm{P}p + \mathrm{Q}q + \mathrm{R}r = \mathrm{S}$ as in art. 14; and suppose any three independent equations of that article, when integrated simultaneously, to be $s = a$, $s' = b$ and $s'' = c$, where s, s', s'' are functions of (x, y, t, z), and a, b, c are the constants introduced by the integration; then shall the primitive be $a = \phi(b, c)$.

For assume $a = \phi(b, c)$ or $s = \phi(s', s'') =$, by substitution, ϕv; then differentiating on the hypothesis that y and z are the only variables, it may be shown that $q = \dfrac{\phi' v \dfrac{dv}{dy} - \dfrac{ds}{dy}}{\dfrac{ds}{dz} - \phi' v \dfrac{dv}{dz}} =$,

by substitution m. Similarly $r = \dfrac{\phi' v \dfrac{dv}{dt} - \dfrac{ds}{dt}}{\dfrac{ds}{dz} - \phi' v \dfrac{dv}{dz}} = n$; where m and n contain the arbitrary function ϕ.

Eliminating p from the proposed equation by means of $dz = pdx + qdy + rdt$, there will result
$\mathrm{S}dx - \mathrm{P}dz - q(\mathrm{Q}dx - \mathrm{P}dy) - r(\mathrm{R}dx - \mathrm{P}dt) = 0$, or
$\mathrm{P}dz = \mathrm{S}dx - m(\mathrm{Q}dx - \mathrm{P}dy) - n(\mathrm{R}dx - \mathrm{P}dt)$.

This satisfies the proposed independently of m and n; for it gives $p = \dfrac{\mathrm{S}}{\mathrm{P}} - \dfrac{m\mathrm{Q}}{\mathrm{P}} - \dfrac{n\mathrm{R}}{\mathrm{P}}$ and $q = m$ and $r = n$; which

substituted in the proposed renders it identical; or $a = \phi(b, c)$ satisfies it independently of the form of ϕ.

These theorems, by means of which the integration of partial is reduced to that of total equations, are due to Lagrange. The reader will find an elaborate chapter on the subject of partial equations in the Calc. des Fonct. Leçon 20.

18. The Theory of Arbitrary Constants when applied to partial equations does not give the same results as in the case of total equations.

The complete primitive of a total equation of the first order contains only one arbitrary constant, because more cannot be eliminated by differentiation; but, since a function of two independent variables, when differentiated partially, gives two fluxional equations, it is manifest that two constants may be eliminated, so that the complete primitive of a partial equation of the first order of three variables necessarily contains *two* arbitrary constants.

Thus, let $z = ax + by$; then $\frac{dz}{dx} = a$ and $\frac{dz}{dy} = b$; and eliminating a, there results $z = x\frac{dz}{dx} + y\frac{dz}{dy}$.

Similarly, if z is a function of n variables, the complete primitive of an equation of the first order shall contain n arbitrary constants.

Hence it appears that any equation is a complete primitive of $z = x\frac{dz}{dx} + y\frac{dz}{dy}$ which contains two arbitrary constants, and which is included in the form $z = x\phi\frac{y}{x}$; which may be called, by way of contradistinction, the *canonical* primitive.

Thus $z = ay + \frac{bx}{y}$ is likewise a complete primitive of the proposed, for it is the same as $z = x\left\{ a\frac{y}{x} + b\frac{x}{y} \right\}$. If this be differentiated partially, and a and b be eliminated, there will result the proposed equation.

If it should happen that both the constants disappear by

one partial differentiation, the equation is not a complete, but a particular primitive.

19. *Given a complete primitive of a partial equation of the first order; required the canonical primitive.*

First, let z be a function of two variables; and let the given primitive be $\text{F}(x, y, z, a, b) = 0 = u$.

Since $b = \phi a$, if $u = 0$ be differentiated with respect to a, there results $0 = \frac{du}{da} + \frac{du}{db}\frac{db}{da} = \frac{du}{da} + \frac{du}{db}\phi' a =$, by substitution, $\text{A} + \text{B}\phi' a$.

If the fluxional equation is of the first degree, the given primitive is of only one dimension with respect to the arbitrary constants; and eliminating them by means of the given primitive and of $\left.\begin{matrix} \phi a = b \\ \text{A} + \text{B}\phi' a = 0 \end{matrix}\right\}$, there will result an equation between the variables in which ϕ remains; which is therefore the canonical primitive.

Next, let $z = f(x, y, t)$; and let the given primitive be $\text{F}(x, y, t, z, a, b, c) = 0 = u$; then, since $c = \phi(a, b) =$, by substitution, ϕv, we have by differentiation,

$$0 = \frac{du}{da}da + \frac{du}{db}db + \frac{du}{dc}dc.$$

$$= \text{A}da + \text{B}db + \text{C}\phi' v \left\{ \frac{dv}{da}da + \frac{dv}{db}db \right\}$$

$$= \left\{ \text{A} + \text{C}\phi' v \frac{dv}{da} \right\} da + \left\{ \text{B} + \text{C}\phi' v \frac{dv}{db} \right\} db.$$

This, on account of the independence of a and b, gives $\text{A} + \text{C}\phi' v \frac{dv}{da} = 0$ and $\text{B} + \text{C}\phi' v \frac{dv}{db} = 0$; and these, combined with $c = \phi(a, b)$ and the given primitive will give by elimination the canonical primitive when the proposed equation is of the first degree.

The converse problem does not admit of a solution, since there is an infinite number of complete primitives belonging to the same canonical primitive.

Ex. 1. $\frac{dz}{dx} = n\frac{dz}{dy}$, of which $z = a + b(nx + y)$ is a complete primitive.

Since $a + \phi a(nx + y) - z = 0 = u \therefore \cdot \cdot \cdot \cdot \cdot \cdot$

$\frac{du}{da} = 0 = 1 + \phi' a(nx + y) \therefore a = \chi(nx + y) \therefore \cdot \cdot \cdot$

$b = \psi(nx + y) \therefore z$, which $= a + b(nx+y), = \phi(nx+y)$.

The same result may be deduced independently of the article; for, differentiating $z = a + b(nx + y)$ on the hypothesis that a and b are the only variables, $da+(nx+y)db=0$, or $da=-(nx + y)db \therefore -nx + y = \phi' b$, or $b = \chi(nx+y)$ $\therefore a$, which $= \phi b, = \psi(nx + y) \therefore z = \phi(nx + y)$.

Ex. 2. $x\frac{dz}{dx} + y\frac{dz}{dy} = z$, of which $z = ax + by$ is a complete primitive.

Here $ax + y\phi a - z = 0 \therefore x + y\phi' a = 0 \therefore$

$a = \chi\frac{y}{x}$, $b=\psi\frac{y}{x}$ and z, which $= x\left\{a + \frac{by}{x}\right\}, = x\phi\frac{y}{x}$.

Ex. 3. $(zy + a)(x + b) + y = 0$.

Here the given primitive is of the second degree with respect to the arbitrary constants, and we cannot eliminate them by means of the three equations. In this case the canonical primitive consists of

$\left.\begin{array}{l}(zy + a)(x + \phi a) + y \quad = 0 \\ (x \ + \phi a) \ + (zy + a)\phi' a = 0\end{array}\right\}$; in which, assigning to ϕ any form whatever, a may be eliminated and the primitive obtained.

20. *Required to find a singular solution when a complete primitive is given.*

Let the complete primitive be $z = f(x, y, a, b)$; and when differentiated on the hypothesis that a and b vary, let the result be $dz = pdx + qdy + \text{A}da + \text{B}db$, where p and q are not affected by the hypothesis. Then, it may be shown, as in ch. 5, 11, that the values of a and b corresponding to a singular solution are to be found from the equation $\text{A}da + \text{B}db = 0$. From this we obtain $\text{A} = 0$ and $\text{B} = 0$. Find then the values of a and b from the two equations $\text{A} = 0$, $\text{B} = 0$; substitute them in the given primitive, and the result is a singular solution.

21. *Partial equations of a higher degree than the first.*

The integration of these depends upon the following formulæ.

$dz = pdx + qdy$; wherefore, integrating by parts,
$z = px + \int(qdy - xdp)$ (1)
$z = qy + \int(pdx - ydq)$ (2)
$z = px + qy - \int(xdp + ydq)$ (3).

22. *Required to integrate the form* $\frac{dz}{dy} = f\,\frac{dz}{dx}$.

Here $q = fp$; and by substitution in (3),
$z = xp + yfp - \int(xdp + yf'pdp)$; and differentiating,
$d(xp + yfp - z) = (x + yf'p)dp$; and since the first member of this equation is an exact fluxion, it follows that the second is also an exact fluxion; and consequently $x + yf'p = \phi'p$ and $xp + yfp - z = \phi p$; in which equations, if any form whatever be assigned to ϕ, and p be eliminated, there will result a primitive of the proposed equation.

Ex. 1. $\frac{dz}{dx}\,\frac{dz}{dy} = 1$.

Here $q = \frac{1}{p}$ $\therefore z$, which

$$= xp + qy - \int(xdp + ydq), = xp + \frac{y}{p} - \int\left\{ xdp - \frac{ydp}{p^2} \right\};\text{ or}$$

$$d\left\{ z - xp - \frac{y}{p} \right\} = \left\{ \frac{y}{p^2} - x \right\} dp \;\therefore\; \frac{y}{p^2} - x = \phi'p \quad (1)$$

and $z - xp - \frac{y}{p} = \phi p$ (2).

Since ϕ is an arbitrary function, suppose $\phi p = a$; then $\phi'p = 0$; and eliminating p from (1) and (2), there results $z + a = 2\sqrt{xy}$; which is only a particular primitive. If $\phi p = 0$, then $\phi'p = 0$, and the primitive is $z^2 = 4xy$.

Ex. 2. $z = x\frac{dz}{dx} + y\frac{dz}{dy} + f\left\{ \frac{dz}{dx},\ \frac{dz}{dy} \right\}$.

$z = px + qy + fv$, where $fv = f(p, q)$; $\therefore$ differentiating,

$$0 = xdp + ydq + f'v\left\{ \frac{dv}{dp}dp + \frac{dv}{dq}dq \right\} \quad . \quad . \quad . \quad . \quad . \quad .$$

$$= \left\{ x + \frac{dv}{dp}f'v \right\} dp + \left\{ y + \frac{dv}{dq}f'v \right\} dq.$$

This is satisfied either by

$\left.\begin{matrix} dp = 0 \\ dq = 0 \end{matrix}\right\}$, or by $\left\{\begin{matrix} x + \frac{dv}{dp} f'v = 0 \\ y + \frac{dv}{dq} f'v = 0 \end{matrix}\right\}$. The first system gives $p = a$, $q = b$: or $z = ax + by + f(a, b)$ is a complete primitive. The second, which is the same as the equations $\text{A} = 0$, $\text{B} = 0$ of art. 20, gives the singular solution.

23. *Required to integrate* $\text{F}\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) = 0$.

Let it be under the form $q = f(x, y, z, p)$; then, since z is to be a function of (x, y), $dz = pdx + qdy$; and by the equation of condition (3) ch. 7, 4, we have

$$\frac{dq}{dx} - \frac{dp}{dy} + p\frac{dq}{dz} - q\frac{dp}{dz} = 0 \quad (1).$$

Differentiating $q = f(x, y, z, p)$, let $dq = \text{D}dp + \text{A}dx + \text{B}dy + \text{C}dz$; wherefore

$\frac{dq}{dx} = \text{D}\frac{dp}{dx} + \text{A}$, and $\frac{dq}{dz} = \text{D}\frac{dp}{dz} + \text{C}$; which, substituted in (1), give

$$\text{D}\frac{dp}{dx} - \frac{dp}{dy} + (p\text{D} - q)\frac{dp}{dz} = -(\text{A} + p\text{C}) \quad (2).$$

This, if q be eliminated by means of $q = f(x, y, z, p)$, is a partial equation, in which p is a function of three variables (x, y, z); and the equations (α), (β), (γ) of art. 14, become

$$\left.\begin{matrix} dx + \text{D}dy = 0 \\ (p\text{D} - q)dx - \text{D}dz = 0 \\ (\text{A} + p\text{C})dx + \text{D}dp = 0 \end{matrix}\right\} \quad (3).$$

Let the primitives of (3), when integrated simultaneously, be $s = a$, $s' = b$, $s'' = c$; then the canonical primitive of (2) is $s = \phi(s', s'')$.

This is a functional equation between x, y, z and p. Take any particular value of p which satisfies it containing an arbitrary constant; call it (p). Let (q) be the corresponding value of q; then, integrating $dz = (p)dx + (q)dy$ as a total equation between three variables, a complete primitive of the proposed equation will be found, from which the canonical primitive may be obtained as in art. 19.

But it is not necessary to integrate the equation (1); for we shall obtain a complete and consequently the canonical primitive, if by means of any of the equations (3),

or of their combinations we can find a value of p containing an arbitrary constant.

It is worthy of remark that if D be eliminated from the two first of equations (3), there results $dz - pdx - qdy = 0$; which may therefore be substituted for either of those equations.

If the proposed equation be under the form $p = f(x, y, z, q)$, the method of integration will be precisely the same, interchanging x and y.

For the case in which z is a function of *three* variables, vid. Lacroix, p. 567.

Ex. 1. $\frac{dz}{dy} - \left\{ x\frac{dz}{dx} + z \right\}^2 = 0$, or $q = (px + z)^2$.

Here $dq = 2(px + z)(xdp + pdx + dz) \therefore$
$\text{D} = 2x(px + z)$; $\text{A} = 2p(px + z)$; $\text{B} = 0$; and
$\text{C} = 2(px + z)$; and the third equation of (3) becomes $2pdx + xdp = 0$; and integrating, $px^2 = a \therefore$

$$(p) = \frac{a}{x^2} \text{ and } (q) = \left(\frac{a}{x} + z\right)^2 \therefore dz = \frac{adx}{x^2} + \left(\frac{a}{x} + z\right)^2 dy$$

$$\text{or } \frac{dz - \frac{adx}{x^2}}{\left(\frac{a}{x} + z\right)^2} = dy \therefore \text{ integrating } y + b + \frac{1}{\frac{a}{x} + z} = 0, \text{ or}$$

$(xz + a)(y + b) + x = 0$. From this the canonical primitive is to be obtained as in art. 19, ex. 3.

Ex. 2. $y\frac{dz}{dx}\frac{dz}{dy} = x\frac{dz^2}{dx^2} + y$.

Here $q = \frac{xp}{y} + \frac{1}{p} \therefore$

$$dq = \left\{ \frac{x}{y} - \frac{1}{p^2} \right\} dp + \frac{p}{y}dx - \frac{xp}{y^2}dy, \text{ or } \text{D} = \frac{xp^2 - y}{yp^2};$$

$$\text{A} = \frac{p}{y}; \ \text{B} = -\frac{xp}{y^2} \text{ and } \text{C} = 0.$$

Eliminating dx from the first and third of equations (3), there results $(\text{A} + p\text{C})dy = dp$, or $\frac{pdy}{y} = dp \therefore$ integrating,

$$p = ay \therefore q \text{ which } = \frac{xp}{y} + \frac{1}{p}, = ax + \frac{1}{ay} \therefore$$

$dz = aydx + axdy + \frac{dy}{ay}$; and again integrating,

$z = axy + \frac{1}{a}ly + b$, which is a complete primitive of the proposed.

Cor. This method will always succeed when the proposed form is linear with respect to either x or y, and z does not enter.

In the following articles we shall apply this method to certain forms which become integrable in consequence of one of the equations (3) or their combinations containing only two variables. Lacroix has given an analytical investigation of all the integrable forms of the first order. (Tome ii. p. 550.) The form may be considered as integrated when a complete primitive has been found; since the canonical primitive may be readily deduced by means of art. 19.

24. *Required to integrate* $\mathrm{F}(p, q, z) = 0$.

Here $q = f(p, z)$; and differentiating, $dq = \mathrm{D}dp + \mathrm{c}dz$, where D and C are functions of (p, z). Also $\mathrm{A} = \mathrm{B} = 0$; whence the equations (3) become

$$\left.\begin{aligned} dx + \mathrm{D}dy &= 0 \\ (p\mathrm{D} - q)dx - \mathrm{D}dz &= 0 \\ p\mathrm{c}dx + \mathrm{D}dp &= 0 \end{aligned}\right\}.$$

Eliminating dx from the second and third, there results $(p\mathrm{D} - q)dp + p\mathrm{c}dz = 0$ (α).

Since this is an equation between p and z, a value of p containing an arbitrary constant, and consequently a complete, and also a canonical primitive of the proposed equation may be obtained.

Ex. $z = \frac{dz}{dx}\frac{dz}{dy}$.

Here $q = \frac{z}{p}$ $\therefore dq = -\frac{zdp}{p^2} + \frac{dz}{p}$, or $\mathrm{D} = -\frac{z}{p^2}$ and $\mathrm{c} = \frac{1}{p}$; $\therefore (\alpha)$ is $-\frac{2zdp}{p} + dz = 0$; $\therefore$ integrating,

$p = \frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ $\therefore (q) = a^{\frac{1}{2}}z^{\frac{1}{2}}$ $\therefore dz = pdx + qdy$, ...

$= \frac{z^{\frac{1}{2}}dx}{a^{\frac{1}{2}}} + a^{\frac{1}{2}}z^{\frac{1}{2}}dy$, or $\frac{dz}{z^{\frac{1}{2}}} = \frac{dx}{a^{\frac{1}{2}}} + a^{\frac{1}{2}}dy$; $\therefore$ a complete primitive is $2z^{\frac{1}{2}} = a^{-\frac{1}{2}}x + a^{\frac{1}{2}}y + b$.

The canonical primitive consists of

$$\left.\begin{aligned} 2a^{\frac{1}{2}}z^{\frac{1}{2}} &= x + ay + \phi a \\ a^{-\frac{1}{2}}z^{\frac{1}{2}} - y &= \phi' a \end{aligned}\right\}.$$

If any form whatever be assigned to ϕ, a may be eliminated and the primitive obtained.

Thus, if we suppose $\phi a = Aa + B$; then $\phi' a = A$, and the primitive is $z = (y + A)(x + B)$.

25. *Required to integrate* $P = Q$, *where* $P = F\left(x, \frac{dz}{dx}\right)$ *and* $Q = f\left(y, \frac{dz}{dy}\right)$.

Differentiating, $\frac{dP}{dx}dx + \frac{dP}{dp}dp = \frac{dQ}{dy}dy + \frac{dQ}{dq}dq$, or

$$dq = \frac{\frac{dP}{dp}dp + \frac{dP}{dx}dx - \frac{dQ}{dy}dy}{\frac{dQ}{dq}}.$$

Assume as before $dq = Ddp + Adx + Bdy + Cdz$; then $D = \frac{dP}{dp} \div \frac{dQ}{dq}$; $A = \frac{dP}{dx} \div \frac{dQ}{dq}$; and $C = 0$.

Substituting these in the last of equations (3), there will result $\frac{dP}{dx}dx + \frac{dP}{dp}dp = 0$; wherefore $P = a$, an arbitrary constant. Assume then $F(x, p) = f(y, q) = a$; and solving these equations, let $p = \chi(a, x)$, $q = \psi(a, y)$; where χ and ψ are known forms; then dz, which $= pdx + qdy$, $= \chi(a, x)dx + \psi(a, y)dy$ (1).

Let $v = \int \chi(a, x)dx$ and $w = \int \psi(a, y)dy$; then, since a is the constant, the correction is in each case an arbitrary function of a, and incorporating the two corrections, there results $z = v + w + \phi a$ (2).

Also from equation (1), $\frac{dz}{da} = 0$; wherefore the canonical

primitive is $z = v + w + \phi a$

$$0 = \frac{dv}{da} + \frac{dw}{da} + \phi' a \Big\}$$; from which a is to be eliminated, assigning any form whatever to ϕ.

Ex. $\frac{b^2 dz}{dx}\,\frac{dz}{dy} = x^2 y^2$.

Assume $\frac{b^2 p}{x^2} = \frac{y^2}{q} = a$; then $p = \frac{ax^2}{b^2}$ and $q = \frac{y^2}{a}$ $\therefore$

$dz = \frac{ax^2 dx}{b^2} + \frac{y^2 dy}{a}$ $\therefore$ the primitive is

$$z = \frac{ax^3}{3b^2} + \frac{y^3}{3a} + \varphi a$$
$$0 = \frac{x^3}{3b^2} - \frac{y^3}{3a^2} + \varphi' a \Big\}.$$

If we suppose $\varphi a = 0$, then $\phi' a = 0$; and eliminating a, there results $3bz = 2x^{\frac{3}{2}} y^{\frac{3}{2}}$.

26. *Required to integrate* $q = p^n \mathrm{XYZ}$.

Here $dq = np^{n-1}\mathrm{XYZ}dp + p^n \mathrm{YZ}d\mathrm{X} + p^n \mathrm{XZ}d\mathrm{Y} + p^n \mathrm{XY}d\mathrm{Z}$; wherefore $\mathrm{D} = np^{n-1}\mathrm{XYZ}$, $\mathrm{A} = p^n \mathrm{YZ}\,\frac{d\mathrm{X}}{dx}$ and $\mathrm{C} = p^n \mathrm{XY}\,\frac{d\mathrm{Z}}{dz}$; and by substitution in the second and third of equations (3), there will result

$$(n-1)pdx - ndz = 0$$
$$p\left\{z\frac{d\mathrm{X}}{dx} + p\mathrm{X}\frac{d\mathrm{Z}}{dz}\right\}dx + n\mathrm{XZ}dp = 0\Big\}.$$ Substituting in the second of these for $p\mathrm{X}\frac{d\mathrm{Z}}{dz}dx$ its value deduced from the first, viz. $\frac{n}{n-1}\mathrm{X}dz$, there results,

$$p\left\{z d\mathrm{X} + \frac{n}{n-1}\mathrm{X}dz\right\} + n\mathrm{XZ}dp = 0, \text{ or}$$

$\frac{dp}{p} + \frac{d\mathrm{X}}{n\mathrm{X}} + \frac{dz}{(n-1)z} = 0$; wherefore integrating,

$p\mathrm{X}^{\frac{1}{n}} z^{\frac{1}{n-1}} = a$, or $p = a\mathrm{X}^{-\frac{1}{n}} z^{-\frac{1}{n-1}}$ and $q = a^n \mathrm{YZ}^{-\frac{1}{n-1}}$.

Also $dz = pdx + qdy = a\mathrm{X}^{-\frac{1}{n}}\mathrm{Z}^{-\frac{1}{n-1}}dx + a^n\mathrm{Y}\mathrm{Z}^{-\frac{1}{n-1}}dy$,

or $\mathrm{Z}^{\frac{1}{n-1}}dz = a\mathrm{X}^{-\frac{1}{n}}dx + a^n\mathrm{Y}dy$; and the required primitive is

$$\left.\begin{aligned}\int \mathrm{Z}^{\frac{1}{n-1}}dz - a\int \mathrm{X}^{-\frac{1}{n}}dx - a^n\int \mathrm{Y}dy &= \varphi a\\ -\int \mathrm{X}^{-\frac{1}{n}}dx - na^{n-1}\int \mathrm{Y}dy &= \varphi' a\end{aligned}\right\}.$$

EQUATIONS OF THE SECOND ORDER.

27. In a partial equation of the second order the same quantities enter as in one of the first; and also, if z be a function of only two variables, $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dxdy}$, $\frac{d^2z}{dy^2}$; for which we shall substitute r, s, t.

The canonical form is $\mathrm{F}(x, y, z, p, q, r, s, t) = 0$. We shall begin with the simplest examples; those, namely, which are of only one dimension with respect to the fluxional coefficients.

28. In order to eliminate n arbitrary functions, the resulting fluxional equation must be at least of the nth order; conversely, the canonical primitive of an equation of the nth order contains n arbitrary functions. It may contain more; for it sometimes requires a higher order of differentiation to eliminate the functions, as in $z = \varphi(x + y) + xy\psi(x - y)$. Vid. Lacroix, tom. i. 78.

29. $\frac{d^2z}{dx^2} = 0$.

If $z = f(x, y)$, we have, integrating, $\frac{dz}{dx} = \varphi y$, or $(dz) = dx\varphi y$; wherefore, again integrating, $z = x\varphi y + \chi y$.

30. $\frac{d^2z}{dx^2} = \mathrm{Q} = f(x, y)$.

If we reverse the operations which the symbols indicate, we have $\frac{dz}{dx} = \int \mathrm{Q}dx + \varphi y$ and $z = \int dx \int \mathrm{Q}dx + x\varphi y + \chi y$.

Similarly, if $\frac{d^2z}{dy^2} = \mathrm{Q}$; $z = \int dy \int \mathrm{Q}dy + y\varphi x + \chi x$.

Cor. When $\mathrm{Q} = 0$, $z = x\varphi y + \chi y$, as before.

Ex. $\frac{ad^2z}{dx^2} = xy$ ∴ integrating, $\frac{adz}{dx} = \frac{x^2y}{2} + \varphi y$ and

$az = \frac{x^3y}{6} + x\varphi y + \chi y.$

31. $\frac{d^2z}{dxdy} = \mathrm{Q} = f(x, y).$

Here $\frac{dz}{dx} = \int \mathrm{Q}dy + \varphi' x$ and $z = \int dx \int \mathrm{Q}dy + \varphi x + \chi y.$

The result will be the same, if we first integrate on the supposition that x is the variable.

Cor. When $\mathrm{Q} = 0$, $z = \varphi x + \chi y.$

Ex. 1. $\frac{d^2z}{dxdy} = x^2y$; ∴ $\frac{dz}{dx} = \frac{x^2y^2}{2} + \varphi' x$ and

$z = \frac{x^3y^2}{6} + \phi x + \chi y.$

Ex. 2. $\frac{d^2z^2}{dx^2dy^2} = x + y + z.$

Substitute $q = \frac{dz}{dy}$; then $\frac{dq^2}{dx^2} = x + y + z$, or

$\frac{dq}{dx} = \sqrt{x + y + z}$ ∴ q or $\frac{dz}{dy} = \frac{2}{3}(x + y + z)^{\frac{3}{2}} + \varphi'(y, z)$

and $z = \frac{2.2}{3.5}(x + y + z)^{\frac{5}{2}} + \varphi(y, z) + \chi(x, z).$

The latter correction is $\chi(x, z)$, because $\int dy(x + y + z)^{\frac{3}{2}}$ and $\int dy\varphi'(y, z)$ must each contain an arbitrary function of (x, z).

Ex. 3. $\frac{d^3z}{dxdydt} = xyt.$

Here $\frac{d^2z}{dxdy} = \int xy.tdt = \frac{xyt^2}{2} + \varphi''(x, y)$ ∴

$\frac{dz}{dx} = \frac{xy^2t^2}{4} + \varphi'(x, y) + \chi'(x, t)$ ∴

$z = \frac{x^2y^2t^2}{8} + \varphi(x, y) + \chi(x, t) + \psi(y, t).$

The general equations, $\frac{d^nz}{dx^n} = \mathrm{P}$; $\frac{d^nz}{dx^{n-1}dy} = \mathrm{Q}$; . . .

$\frac{d^n z}{dx^{n-2}dy^2} = \text{R}$; $\cdots \frac{d^n z}{dy^n} = \text{T}$, where P, Q, R, $\cdots$ T are functions of (x, y, z), are all to be integrated by the above methods.

32. $\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx} = \text{Q}$ *where* P *and* Q *are functions of* (x, y).

Here $r + \text{P}p = \text{Q}$, or $\frac{dp}{dx} + \text{P}p = \text{Q}$; which, if y be considered constant, is linear in p; and consequently, integrating, $p = e^{-\int \text{P}dx}\{\int \text{Q}e^{\int \text{P}dx}dx + \phi y\}$ and

$z = \int e^{-\int \text{P}dx}\{\int \text{Q}e^{\int \text{P}dx}dx + \phi y\}dx + \chi y$.

Similarly, if $\frac{d^2z}{dy^2} + \text{P}\frac{dz}{dy} = \text{Q}$;

$z = \int e^{-\int \text{P}dy}\{\int \text{Q}e^{\int \text{P}dy}dy + \phi x\}dy + \chi x$.

Cor. When P $= 0$, $z = \int dx \int \text{Q}dx + x\phi y + \chi y$ as in art. 29.

33. $\frac{d^2z}{dxdy} + \text{P}\frac{dz}{dx} = \text{Q}$ *where* P *and* Q *are functions of* (x, y).

Here it may be shown that

$z = \int e^{-\int \text{P}dy}\{\int \text{Q}e^{\int \text{P}dy}dy + \phi' x\}dx + \chi y$.

Similarly, if $\frac{d^2z}{dxdy} + \text{P}\frac{dz}{dy} = \text{Q}$;

$z = \int e^{-\int \text{P}dx}\{\int \text{Q}e^{\int \text{P}dx}dx + \phi' y\}dy + \chi x$.

Cor. When P $= 0$, $z = \int dx \int \text{Q}dy + \phi x + \chi y$ as in art. 31.

34. EXAMPLES.

Ex. 1. $xy\frac{d^2z}{dx^2} + ny\frac{dz}{dx} = a$.

Here $\frac{dp}{dx} + \frac{np}{x} = \frac{a}{xy}$ $\therefore p = \frac{a}{ny} + \frac{1}{x^n}\varphi y$ $\therefore$

$z = \frac{ax}{ny} - \frac{\varphi y}{(n-1)x^{n-1}} + \chi y$.

Ex. 2. $xy\frac{d^2z}{dxdy} + nx\frac{dz}{dx} = ay$.

Here $\frac{dp}{dy} + \frac{np}{y} = \frac{a}{x} \therefore z = \frac{aylx}{n+1} + \frac{\varphi x}{y^n} + \chi y.$

35. *Required to integrate the form*

$$\text{F}\left\{x, y, z, \frac{dz}{dy}, \frac{d^2z}{dy^2} \cdots \frac{d^nz}{dy^n}\right\} = 0.$$

If x be considered constant, this is a fluxional equation between two variables, and is to be integrated as in the preceding chapter. Its complete primitive contains n arbitrary constants, *i. e.* n different arbitrary functions of x.

36. *An equation of the* $m + n$*th order of the form*

$$\text{F}\left\{x, y, \frac{d^nz}{dy^n}, \frac{d^{n+1}z}{dy^n dx}, \frac{d^{n+2}z}{dy^n dx^2} \cdots \frac{d^{n+m}}{dy^n dx^m}\right\} = 0 \text{ } \textit{may be reduced to one of the } n\textit{th order.}$$

For substitute $v = \frac{d^nz}{dy^n}$, and the proposed form becomes

$$\text{F}\left\{x, y, v, \frac{dv}{dx}, \frac{d^2v}{dx^2} \cdots \frac{d^mv}{dx^m}\right\} = v,$$ which is of the mth order.

This, if y be considered constant, is to be integrated as a fluxional equation between two variables; and its complete primitive contains m arbitrary functions of y. Having thus found v, and integrating $\frac{d^nz}{dy^n} = v$, x being constant, the final equation or required primitive will contain m arbitrary functions of y and n arbitrary functions of x.

37. *Required to integrate* $\frac{d^{n+m}z}{dx^n dy^m} + \text{P}\frac{d^mz}{dy^m} = \text{Q}$ *where* P *and* Q *are functions of* (x, y).

Substituting $v = \frac{d^mz}{dy^m}$, the proposed becomes

$\frac{d^nv}{dx^n} + \text{P}v = \text{Q}$; which, when integrated, will contain n arbitrary functions of y; and the required primitive shall contain altogether $m + n$ arbitrary functions.

38. The propositions of the preceding articles may be

readily extended to equations in which z is a function of three or more variables.

For let the form be $\text{F}\left\{x, y, t, z, \frac{dz}{dt}, \frac{d^2z}{dt^2} \cdots \frac{d^nz}{dt^n}\right\} = 0.$

This is to be integrated as if z and t were the only variables, and its primitive will contain n arbitrary functions of (x, y).

The form $\text{F}\left\{x, y, t, \frac{d^{n+p}z}{dy^n dx^p}, \frac{d^{n+p+1}z}{dt dy^n dx^p} \cdots \frac{d^{n+m+p}z}{dt^m dy^n dx^p}\right\} = 0,$

by substituting $v = \frac{d^{n+p}z}{dy^n dx^p}$ is reduced to

$$\text{F}\left\{x, y, t, v, \frac{dv}{dt} \cdots \frac{d^mv}{dt^m}\right\} = 0.$$

Proceeding as in art. 36, the final equation will contain $n + m + p$ arbitrary functions; viz. m of (x, y), n of (t, y), and p of (t, x).

39. We shall now consider equations, which, though of the first degree, are of more than one dimension with respect to the fluxional coefficients.

We shall have to refer to the following equations:

$dz = pdx + qdy$

dp, which $= \frac{dp}{dx}dx + \frac{dp}{dy}dy$, $= rdx + sdy$

$dq = sdx + tdy$ and $d^2z = rdx^2 + 2sdxdy + tdy^2$.

40. $\text{R}\frac{d^2z}{dx^2} + \text{S}\frac{d^2z}{dxdy} + \text{T}\frac{d^2z}{dy^2} = \text{V}$ *where* R, S, T *and* V *are each functions of* $\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right)$.

Eliminating by means of the equations

$$\left.\begin{aligned} \text{R}r + \text{S}s + \text{T}t &= \text{V} \\ dp &= rdx + sdy \\ dq &= sdx + tdy \end{aligned}\right\}$$ any two of the fluxional coefficients as r and t, we have $\text{R}\frac{dp - sdy}{dx} + \text{S}s + \text{T}\frac{dq - sdx}{dy} = \text{V}$, or

$$\text{R}dpdy + \text{T}dqdx - \text{V}dxdy - s(\text{R}dy^2 - \text{S}dxdy + \text{T}dx^2) = 0 \quad (1).$$

Since s is indeterminate, this is satisfied by the system

$$\left.\begin{aligned} \text{R}dy^2 - \text{S}dxdy + \text{T}dx^2 &= 0 \quad (2) \\ \text{R}dpdy + \text{T}dqdx - \text{V}dxdy &= 0 \quad (3) \end{aligned}\right\}.$$

The equation (2), which is the same whichever of the

fluxional coefficients be eliminated, is of the second degree with respect to $\frac{dy}{dx}$; and consequently (ch. 5, 1) is to be resolved into the two, $dy - kdx = 0$, $dy - k'dx = 0$, where k and k' are the roots of $\frac{dy^2}{dx^2} - \frac{\text{S}}{\text{R}}\frac{dy}{dx} + \frac{\text{T}}{\text{R}} = 0$.

Hence, if to these we add $dz = pdx + qdy$, we shall have four equations which obtain at the same time between the five quantities x, y, z, p and q, by whose integration, a first primitive is to be found. But the resulting form of the primitive being inconvenient, the following theorem has been proposed and demonstrated by Monge.

Substituting the two values of dy in equation (3), we have the two systems

$$\left.\begin{aligned} dy - kdx &= 0 \\ \text{R}kdp + \text{T}dq - \text{V}kdx &= 0 \end{aligned}\right\} (4) \text{ and}$$

$$\left.\begin{aligned} dy - k'dx &= 0 \\ \text{R}k'dp + \text{T}dq - \text{V}k'dx &= 0 \end{aligned}\right\} (5).$$

Suppose that by integrating (4) we obtain $\left.\begin{aligned} \text{M} &= a \\ \text{N} &= b \end{aligned}\right\}$; then shall one of the first primitives be $a = \varphi b$.

For assume $a = \varphi b$; and let

$da = \text{A}dx + \text{B}dy + \text{C}dz + \text{D}dp + \text{E}dq = 0$

$db = \text{A}'dx + \text{B}'dy + \text{C}'dz + \text{D}'dp + \text{E}'dq = 0.$

In these equations, substitute $dy = kdx$;

$dz = pdx + qdy = (p + qk)dx$; and $dq = \frac{\text{V}kdx - \text{R}kdp}{\text{T}}$;

and they become

$$\left.\begin{aligned} &\left\{\text{A} + \text{B}k + \text{C}(p + qk) + \frac{\text{E}\text{V}k}{\text{T}}\right\}dx + \left\{\text{D} - \frac{\text{E}\text{R}k}{\text{T}}\right\}dp = 0 \\ &\left\{\text{A}' + \text{B}'k + \text{C}'(p + qk) + \frac{\text{E}'\text{V}k}{\text{T}}\right\}dx + \left\{\text{D}' - \frac{\text{E}'\text{R}k}{\text{T}}\right\}dp = 0 \end{aligned}\right\}.$$

These, since dx and dp are indeterminate, give the following,

$$\left.\begin{aligned} \text{A} + \text{B}k + \text{C}(p + qk) + \frac{\text{E}\text{V}k}{\text{T}} &= 0 \\ \text{A}' + \text{B}'k + \text{C}'(p + qk) + \frac{\text{E}'\text{V}k}{\text{T}} &= 0 \end{aligned}\right\} \text{ and } \left.\begin{aligned} \text{D} - \frac{\text{E}\text{R}k}{\text{T}} &= 0 \\ \text{D}' - \frac{\text{E}'\text{R}k}{\text{T}} &= 0 \end{aligned}\right\}.$$

Also (ex hyp.) $da = \varphi' bdb$; or

$\text{A}dx + \text{B}dy + \text{C}dz + \text{D}dp + \text{E}dq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$
$= \varphi' b(\text{A}'dx + \text{B}'dy + \text{C}'dz + \text{D}'dp + \text{E}'dq)$; in which substitute $dz = pdx + qdy$, and there results
$(\text{A} + \text{C}p)dx + (\text{B} + \text{C}q)dy + \text{D}dp + \text{E}dq \quad . \quad . \quad . \quad . \quad . \quad .$
$= \varphi' b((\text{A}' + \text{C}'p)dx + (\text{B}' + \text{C}'q)dy + \text{D}'dp + \text{E}'dq)$.

By means of this and the four preceding equations, eliminating A, A′, D, D′, there will result

$$(\text{B} + \text{C}q)(dy - kdx) + \frac{\text{E}}{\text{T}}(\text{R}kdp + \text{T}dq - \text{V}kdx) \quad . \quad . \quad . \quad .$$

$$= \varphi' b \left\{ (\text{B}' + \text{C}'q)(dy - kdx) + \frac{\text{E}'}{\text{T}}(\text{R}kdp + \text{T}dq - \text{V}kdx) \right\};$$

which is the same as $\text{R}kdp + \text{T}dq - \text{V}kdx = m(dy - kdx)$, if

$$m = -\frac{\text{B} + \text{C}q - \varphi' b(\text{B}' + \text{C}'q)}{\frac{1}{\text{T}}(\text{E} - \varphi' b.\text{E}')}, \text{ where } m \text{ contains } \varphi.$$

This last equation satisfies the proposed independently of m; for substituting in it $dp = rdx + sdy$ and $dq = sdx + tdy$, it becomes $(\text{R}kr + \text{T}s - \text{V}k + mk)dx + (\text{R}ks + \text{T}t - m)dy = 0$; which, on account of the independence of dx and dy, gives

$$\left.\begin{aligned} \text{R}kr + \text{T}s - \text{V}k + mk &= 0 \\ \text{R}ks + \text{T}t - m &= 0 \end{aligned}\right\};$$

from which, eliminating m, we have $\text{R}kr + \text{R}k^2s + \text{T}s + \text{T}kt - \text{V}k = 0$; and lastly, eliminating $\text{R}k$ from this by means of (2) $\text{R}k^2 - sk + \text{T} = 0$, there will result $k(\text{R}r + ss + \text{T}t - \text{V}) = 0$, or $\text{R}r + ss + \text{T}t - \text{V} = 0$, the proposed equation.

Similarly, if by integrating (5) we obtain $\left.\begin{aligned} \text{M}' &= a' \\ \text{N}' &= b' \end{aligned}\right\}$, the other first primitive shall be $a' = \chi' b$.

Having obtained the two first primitives, if we can integrate either of them, the result will be the canonical fluent of the proposed equation.

41. It may be observed that the equation $dz = pdx + qdy$ belongs to both the systems (4) and (5); and we are at liberty to combine the three equations in each in any way whatever so as to obtain $\left.\begin{aligned} \text{M} &= a \\ \text{N} &= b \end{aligned}\right\}$.

Since the three equations contain five variables x, y, z, p and q, we can eliminate only two by their combination, and one of the resulting equations will in consequence contain three variables.

This may not be independently integrable (ch. 7, 4); but

we are not to conclude that in this case the fluent of the proposed cannot be a single equation. We shall illustrate these principles by particular examples.

42. $\frac{d^2z}{dx^2} + \text{A}\frac{d^2z}{dxdy} + \text{B}\frac{d^2z}{dy^2} = \text{C}.$

Let k and k' be the roots of $k^2 - \text{A}k + \text{B} = 0$; then the system (4) is $\left.\begin{aligned} dy - kdx &= 0 \\ kdp + \text{B}dq - \text{C}kdx &= 0 \end{aligned}\right\}$ which, since $\text{B} = kk'$, is the same as $\left.\begin{aligned} dy - kdx &= 0 \\ dp + k'dq - \text{C}dx &= 0 \end{aligned}\right\}$.

Hence the two first primitives are

$\left.\begin{aligned} p + k'q - \text{C}x &= \varphi'(y - kx) \quad (1) \\ p + kq - \text{C}x &= \psi'(y - k'x) \quad (2) \end{aligned}\right\}$.

To integrate (1) as in art. 9, we have $p + k'q = \text{C}x + \varphi'(y - kx)$, and consequently the equations (α) and (β) of that article become $dy - k'dx = 0 = ds$ and $dz - (\text{C}x + \varphi'(y - kx))dx = 0 = ds'$; and integrating the first, $y = s + k'x$; which reduces the second to $ds' = dz - \text{C}xdx - \varphi'(s + (k' - k)x)dx$; and integrating, $s' = z - \frac{\text{C}x^2}{2} - \phi(y - kx)$; where the constant coefficient $k' - k$ is omitted as not affecting the form of ϕ, and s is replaced by its value $y - k'x$. Hence the required fluent is $z = \frac{\text{C}x^2}{2} + \varphi(y - kx) + \psi(y - k'x)$; which is the same that would result from the integration of (2).

Or thus: Find the values of p and q from the equations (1) and (2).

Substituting them in $dz = pdx + qdy$, we shall have

$$dz = \text{C}xdx + \frac{1}{k(k'-k)}\Big\{(dy - kdx)\varphi'(y - kx) \quad \cdots$$

$$- (dy - k'dx)\psi'(y - k'x)\Big\}, \text{ and integrating}$$

$$z = \frac{\text{C}x^2}{2} + \phi(y - kx) + \psi(y - k'x).$$

43. $\frac{d^2z}{dx^2} + \text{A}\frac{d^2z}{dxdy} + \text{B}\frac{d^2z}{dy^2} = \text{P}$ *where* $\text{P} = f(x, y)$.

Here the system (4) is the same as $\left.\begin{aligned} dy - kdx &= 0 \\ dp + k'dq - \text{P}dx &= 0 \end{aligned}\right\}$;

and one of the first primitives is
$p + k'q = \int \text{P}'dx + \phi'(y - kx)$ (1), where P is accented to show that previous to the integration, y must be eliminated from it by means of $y - kx = s$; and after the integration, y is to be restored by means of the same equation.

To integrate (1) we have from art. 9. the system

$$\left.\begin{aligned} dy - k'dx &= 0 \quad (\alpha) \\ dz - \{\int \text{P}'dx + \phi'(y - kx)\}dx &= 0 \quad (\beta) \end{aligned}\right\}.$$

Eliminating y from (β) by means of $y - k'x = s'$, let it become $dz = dx\int \text{P}''dx + \phi'(s' + (k' - k)x)dx$; then, integrating, $z = \int dx \int \text{P}''dx + \varphi(y - kx) + \psi(y - k'x)$; where the constant coefficient is neglected as not affecting the form of φ; s' is replaced by its value $y - k'x$ and the double accent on P denotes that previous to the first integration y is to be eliminated by means of $y - kx = s$; and previous to the second, by means of $y - k'x = s'$, and the values of y restored in each case.

We shall obtain the same result from the system (5).

In integrating simultaneously two independent equations as (α) and (β) which obtain at the same time, we must always by means of (α) previously eliminate from (β) any quantity the fluxion of which enters into (α) but not into (β); for, if we integrate (β) on the supposition that the quantity is constant, our supposition is inconsistent with the hypothesis of (α) (art. 10).

Ex. 1. $\dfrac{d^2z}{dx^2} - \dfrac{2a}{b}\dfrac{d^2z}{dxdy} + \dfrac{a^2}{b^2}\dfrac{d^2z}{dy^2} = 0.$

Here $k = k' = -\dfrac{a}{b}$; and we have only one system,

$$\left.\begin{aligned} dy + \frac{a}{b}dx &= 0 \\ dp - \frac{a}{b}dq &= 0 \end{aligned}\right\}; \text{ and a first primitive is}$$

$bp - aq = \phi'(ax + by)$.

To integrate this as in art. 9, the system is

$$\left.\begin{aligned} adx + bdy &= 0 \\ \phi'(ax + by)dx - bdz &= 0 \end{aligned}\right\} \therefore ax + by = s \therefore$$

$\phi's.dx - bdz = 0$, $\therefore$ integrating, $x\phi's - bz = \psi s$, or
$z = x\varphi(ax + by) + \psi(ax + by)$.

Ex. 2. $\dfrac{d^2z}{dx^2} + \text{A}\dfrac{d^2z}{dxdy} + \text{B}\dfrac{d^2z}{dy^2} = xy.$

Here the system (4) is $\left.\begin{matrix} dy - kdx = 0 \\ dp + k'dq - xydx = 0 \end{matrix}\right\} \therefore y - kx = s \therefore$ $dp + k'dq - x(s + kx)dx = 0 \therefore$ integrating,

$$s' = p + k'q - \frac{sx^2}{2} - \frac{kx^3}{3} = p + k'q - \frac{x^2(3y - kx)}{6} \therefore$$

the two first primitives are

$$\left.\begin{matrix} p + k'q = \frac{x^2(3y - kx)}{6} + \phi'(y - kx) \\ p + kq = \frac{x^2(3y - k'x)}{6} + \psi(y - k'x) \end{matrix}\right\}; \text{ and the required}$$

fluent is $z = \frac{x^3y}{6} - \frac{Ax^4}{4.6} + \phi(y - kx) + \psi(y - k'x)$.

44. $\frac{dz^2}{dy^2}\frac{d^2z}{dx^2} - \frac{2dz}{dx}\frac{dz}{dy}\frac{d^2z}{dxdy} + \frac{dz^2}{dx^2}\frac{d^2z}{dy^2} = 0.$

Since $r - \frac{2p}{q}s + \frac{p^2}{q^2}t = 0 \therefore k = k' = -\frac{p}{q}$; and there is only one system,

$$\left.\begin{matrix} dy + \frac{p}{q}dx = 0 \\ dp - \frac{p}{q}dq = 0 \end{matrix}\right\} \text{or} \quad \left.\begin{matrix} dz = 0 \\ qdp - pdq = 0 \end{matrix}\right\} \therefore \text{integrating,}$$

$$\left.\begin{matrix} z = s \\ \frac{p}{q} = s' \end{matrix}\right\} \therefore \text{a first primitive is } \frac{p}{q} = \phi z.$$

There is no direct method of integrating this; but, since $\frac{p}{q} = -k = -\frac{dy}{dx} \therefore dy = -\phi z dx = -\phi s dx \therefore$ integrating, $y = x\phi s + \psi s = x\phi z + \psi z$, which is the required fluent.

45. $\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} + \frac{4}{x+y}\frac{dz}{dx} = 0.$

Here $r - t = -\frac{4p}{x+y}$; or $k^2 + 1 = 0 \therefore k = 1$ and $k' = -1$; and the two systems are

$$\left.\begin{aligned} dy - dx &= 0 \\ dp - dq + \frac{4pdx}{x+y} &= 0 \end{aligned}\right\}(4) \text{ and } \left.\begin{aligned} dy + dx &= 0 \\ dp + dq + \frac{4pdx}{x+y} &= 0 \end{aligned}\right\}(5)$$

The integration of either of these is not very obvious. The system (5) has the more promising appearance; and beginning with it, we have $y + x = s$, which reduces its conjugate to $dp + dq + \frac{4pdx}{s} = 0$. But applying the test of integrability (ch. 7. 4), it appears that this is not independently integrable; or one of the first primitives is not a single equation.

In system (4), $dp - dq + \frac{4pdx}{x+y} = 0$ is the same as $(x + y)d(p - q) + (p - q)2dx + 2(pdx + qdy)$; and combining it with its conjugate, which gives $dy = dx$, we have $(x + y)d(p - q) + (p - q)(dx + dy) + 2dz = 0$; and integrating $(x + y)(p - q) + 2z = \phi'(y - x)$; which is the other first primitive.

To integrate this, we have the system

$$\left.\begin{aligned} dy + dx &= 0 \\ (x + y)dz - (\phi'(y - x) - 2z)dx &= 0 \end{aligned}\right\} \therefore y + x = s,$$

and $dz + \frac{2zdx}{s} = \frac{\phi'(s-2x)dx}{s}$ $\therefore$ integrating,

$sz = e^{-\frac{2x}{s}}\left\{\psi s + \int e^{\frac{2x}{s}} dx\phi'(s - 2x)\right\}$. This can be integrated only by assigning a form to ϕ; and when the integration has been effected, s is to be changed into $y + x$.

This example shows that the fluent may be expressed as a single equation, even when the equation which leads to one of the first primitives is not independently integrable. The example $\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} = \frac{2}{x}\frac{dz}{dx}$ is still more remarkable. The equations which lead to the two first primitives are neither of them independently integrable; and yet the fluent is $z = \phi(y+x) + \psi(y - x) - x\{\phi'(y + x) - \psi'(y - x)\}$.

If we reverse the process of integration, we shall see the reason of this circumstance; for, eliminating ψs from $sz = e^{-\frac{2x}{s}}\left\{\psi s + \int e^{\frac{2x}{s}} dx\phi'(s - 2x)\right\}$ by means of two partial differentiations, we obtain $(x+y)(p - q) + 2z = \phi'(y-x)$, the original first primitive; but with respect to the function

ϕ, it is so involved that we have not the means of eliminating it.

46. $z\dfrac{d^2z}{dx^2} - \dfrac{dz^2}{dx^2} + \text{A}\left\{ z\dfrac{d^2z}{dxdy} - \dfrac{dz}{dx}\dfrac{dz}{dy} \right\} \quad . \quad . \quad . \quad .$

$$+ \text{B}\left\{ z\frac{d^2z}{dy^2} - \frac{dz^2}{dy^2} \right\} = 0.$$

Here $zr - p^2 + \text{A}(zs - pq) + \text{B}(zt - q^2) = 0$; which is homogeneous with respect to z and the fluxional coefficients.

Substitute $z = e^v$; and let p', q', r', s', t' be the corresponding fluxional coefficients of $v = f(x, y)$; then, differentiating, $p = e^v p'$; $q = e^v q'$; $r = e^v(p'^2 + r')$; $s = e^v(p'q' + s')$ and $t = e^v(q'^2 + t')$; and by substitution, the proposed becomes, dividing by e^{2v}, $r' + \text{A}s' + \text{B}t' = 0$ $\therefore$ art. (40) $v = l.\phi(y - kx) + l.\psi(y - k'x) = l\{\phi(y - kx).\psi(y - k'x)\}$ $\therefore z$, which $= e^v = \phi(y - kx).\psi(y - k'x)$; where k and k' are the roots of $k^2 - \text{A}k + \text{B} = 0$.

47. If the proposed be $\dfrac{d^2z}{dx^2} + a^2\dfrac{d^2z}{dy^2} = 0$, the fluent is $z = \phi(x - ay\sqrt{-1}) + \psi(x + ay\sqrt{-1})$; and to give this a real form, assume

$$\left.\begin{aligned} \phi(x - ay\sqrt{-1}) &= \chi(x - ay\sqrt{-1}) + k\frac{x - ay\sqrt{-1}}{\sqrt{-1}} \\ \psi(x + ay\sqrt{-1}) &= \chi(x + ay\sqrt{-1}) - k\frac{x + ay\sqrt{-1}}{\sqrt{-1}} \end{aligned}\right\};$$

then we have $z = \chi(x - ay\sqrt{-1}) + \chi(x + ay\sqrt{-1}) - 2kay$, which is always real; and it is the canonical fluent, since it contains two arbitrary functions.

48. LINEAR EQUATIONS.

The canonical form is $r + \text{S}s + \text{T}t + \text{P}p + \text{Q}q + \text{M}z = \text{N}$, where S, T $\cdots$ N are functions of (x, y). This is not generally integrable even when S, T $\cdots$ N are constant quantities. We shall not investigate the equations of condition for the general equation, but shall consider only particular forms.

49. *Required to investigate when*

$$\frac{d^2z}{dxdy} + \text{P}\frac{dz}{dx} + \text{Q}\frac{dz}{dy} + \text{M}z = \text{N}$$

is independently integrable.

Since $s = \frac{dp}{dy}$ and $q = \frac{dz}{dy}$, this becomes

$dp + \text{P}p dy + \text{Q}dz + \text{M}z dy = \text{N}dy$, or
$dp + (\text{P}p + \text{M}z - \text{N})dy + \text{Q}dz = 0$.

Applying the equation (γ) of ch. 7. 4, this is independently integrable only when $\text{M} - \frac{d\text{Q}}{dy} - \text{PQ} = 0$, or

$$\text{M} = \text{PQ} + \frac{d\text{Q}}{dy}.$$

Similarly it may be shown that the proposed is independently integrable when $\text{M} = \text{PQ} + \frac{d\text{Q}}{dx}$.

50. *Required to integrate the linear form*

$$\frac{d^2z}{dxdy} + \text{P}\frac{dz}{dx} + \text{Q}\frac{dz}{dy} + \left\{\text{PQ} + \frac{d\text{P}}{dx}\right\}z = \text{V}.$$

Substitute $v = q + \text{P}z$; then $\frac{dv}{dx} = s + \text{P}p + \frac{d\text{P}}{dx}z$, and the proposed becomes $\frac{dv}{dx} + \text{Q}v = \text{V}$; and integrating,

$v = e^{-\int \text{Q}dx}\{\int e^{\int \text{Q}dx}\text{V}dx + \varphi y\}$, which therefore $= \frac{dz}{dy} + \text{P}z$.

This also being linear, it is to be integrated as before and the correction is to be an arbitrary function of x.

Ex. 1. $\frac{d^2z}{dxdy} + a\frac{dz}{dx} + b\frac{dz}{dy} + abz = \text{V}$.

Substitute $v = q + az$; then $\frac{dv}{dx} = s + ap$, $\therefore$

$$\frac{dv}{dx} + bv = \text{V} \therefore v = e^{-bx}\left\{\int e^{bx}\text{V}dx + \psi y\right\} = \frac{dz}{dy} + az \therefore$$

$$z = e^{-ay}\varphi x + e^{-bx}\psi y + e^{-ay}\int e^{ay-bx}dy\int e^{bx}\text{V}dx \quad \cdots\cdots$$
$$= e^{-ay}\varphi x + e^{-bx}\psi y + e^{-ay-bx}\int e^{ay}dy\int e^{bx}\text{V}dx.$$

This example may be also integrated by substituting $v = p + bz$.

Ex. 2. $\frac{d^2z}{dxdy} - \frac{1}{x-y}\left\{\frac{dz}{dx} + \frac{dz}{dy}\right\} = 0$.

Here $\text{PQ} + \frac{d\text{Q}}{dy} = 0$; $\therefore$ substitute $v = p - \frac{z}{x-y}$; and

the proposed becomes $\frac{dv}{dy} - \frac{v}{x-y} = 0$ ∴
$z = (x - y)(\varphi x + \psi y)$.

51. *Required to integrate the linear equation* $\frac{d^2z}{dxdy} + \mathrm{P}\frac{dz}{dx} + \mathrm{Q}\frac{dz}{dy} + \mathrm{M}z = \mathrm{N}$ *when the criterion of integrability is not satisfied.*

Substitute $v = q + \mathrm{P}z$; then it becomes
$\frac{dv}{dx} - \frac{d\mathrm{P}}{dx}z + \mathrm{Q}(v - \mathrm{P}z) + \mathrm{M}z = \mathrm{N}$, or

$$z = \left\{\frac{dv}{dx} + \mathrm{Q}v - \mathrm{N}\right\} \div \left\{\frac{d\mathrm{P}}{dx} + \mathrm{PQ} - \mathrm{M}\right\} \quad (1).$$

From this equation $\frac{dz}{dy}$ or q may be found; and if these values of z and q be substituted in $v = q + \mathrm{P}z$, there will result an equation of the form

$\frac{d^2v}{dxdy} + \mathrm{P}'\frac{dv}{dx} + \mathrm{Q}'\frac{dv}{dy} + \mathrm{M}'v = \mathrm{N}'$; or by substitution,
$s' + \mathrm{P}'p' + \mathrm{Q}'q' + \mathrm{M}'v = \mathrm{N}'$ (2).

If this satisfies the criterion of integrability, *i. e.* if $\mathrm{M}' =$ either $\mathrm{P}'\mathrm{Q}' + \frac{d\mathrm{P}'}{dx}$ or $\mathrm{P}'\mathrm{Q}' + \frac{d\mathrm{Q}'}{dy}$, v may be obtained in terms of two arbitrary functions of (x, y); which, substituted in (1), will give the required fluent.

If (2) is not integrable, it must be again transformed by substituting $v' = q' + \mathrm{P}'v$; and the process must be repeated till we arrive at an equation in which the criterion is satisfied.

Let there be *three* transformations necessary; then we have an equation which, when integrated, expresses v'' in terms of two arbitrary functions of (x, y); and we shall have to deduce successively the values of v', v and z.

This method is due to Laplace; and he has shown in the *Mémoires de l'Académie, ann.* 1773, that it will always effect the integration when the primitive of the proposed can be expressed in finite terms.

Ex. $\frac{d^2z}{dxdy} + \frac{a}{x+y}\frac{dz}{dx} + \frac{b}{x+y}\frac{dz}{dy} + \frac{ab+b-2(a+1)}{(x+y)^2}z = 0.$

Here the criterion is not satisfied; substitute $v = q + \frac{az}{x+y}$;

then

$$\frac{dv}{dx} + \frac{az}{(x+y)^2} + \frac{b}{x+y}\left\{v - \frac{az}{x+y}\right\} + \frac{ab+b-2(a+1)}{(x+y)^2}z = 0,$$

or $mz = (x+y)^2p' + b(x+y)v$ (1), where $m = a-b+2$

$\therefore mq = 2(x+y)p' + (x+y)^2s' + b(x+y)q' + bv$ $\therefore$

mv, which $= mq + \frac{maz}{x+y}$, ...

$= (a+2)(x+y)p' + (x+y)^2s' + b(x+y)q' + (a+1)bv$;

or $s' + \frac{a+2}{x+y}p' + \frac{b}{x+y}q' + \frac{(a+2)(b-1)}{(x+y)^2}v = 0$ (2).

Here the criterion is satisfied; and integrating,

$v = (x+y)^{-a-2}\{\phi x + \int(x+y)^m\psi y.dy\}$.

Having found v, we obtain $\frac{dv}{dx}$, or

$$p' = -(a+2)(x+y)^{-a-3}\{\phi x + \int(x+y)^m\psi y.dy\} \}\ \therefore$$
$$+ (x+y)^{-a-2}\{\phi' x + m\int(x+y)^{m-1}\psi y.dy\} \}$$

substituting in (1),

$$mz = -(a+2)(x+y)^{-a-1}\{\phi x + \int(x+y)^m\psi y.dy\}$$
$$+ (x+y)^{-a}\{\phi' x + m\int(x+y)^{m-1}\psi y.dy\}$$
$$+ b(x+y)^{-a-1}\{\phi x + \int(x+y)^m\psi y.dy\}$$

$$\therefore z = -(x+y)^{-a-1}\left\{\phi x + \int(x+y)^m\psi y.dy\right\} + \frac{(x+y)^{-a}\phi' x}{m}$$

$+ (x+y)^{-a}\int(x+y)^{m-1}\psi y.dy$.

Cor. When $a = b$ or the equation is

$s + \frac{a}{x+y}(p+q) + \frac{a^2-a-2}{(x+y)^2}z = 0$, we have $m = 2$.

But $\int(x+y)^2\psi' y.dy = (x+y)^2\psi y - 2\int(x+y)\chi'' y.dy$,

if $\chi'' y = \psi y$, $= (x+y)^2\psi y - 2(x+y)\chi' y + 2\chi y$.

Similarly $\int(x+y)\psi' y.dy = (x+y)\psi y - \chi' y$, $\therefore$

by substitution,

$$z = \frac{(x+y)^{-a}\phi' x}{2} - (x+y)^{a-1}\phi x + (x+y)^{-a}\chi' y \ . \ . \ .$$
$$- 2(x+y)^{-a-1}\chi y$$
$$= \frac{(x+y)^{-a}}{2}(\phi' x + \chi' y) - (x+y)^{a-1}(\phi x + \chi y).$$

52. Euler integrates the general linear equation by the introduction of v and w, two new functions of x and y; and by making certain assumptions, he transforms the proposed equation in which z is a function of (x, y) into another of the form of the last article in which z is a function of (v, w).

The following equations will be required. We shall substitute p', q', r', s', t' not as before for the fluxional coefficients of $v = f(x, y)$ but for those of $z = \text{F}(v, w)$.

Also substitute $\dfrac{dv}{dx} = m$, $\dfrac{dv}{dy} = n$; $\dfrac{dw}{dx} = m'$, $\dfrac{dw}{dy} = n'$.

Since $\dfrac{dz}{dx} = \dfrac{dz}{dv}\dfrac{dv}{dx} + \dfrac{dz}{dw}\dfrac{dw}{dx}$, we have

$p = mp' + m'q'$. Similarly,
$q = np' + n'q'$

$$r = m^2r' + 2mm's' + m'^2t' + \frac{dm}{dx}p' + \frac{dm'}{dx}q'$$

$$s = mnr' + (mn' + m'n)s' + m'n't' + \frac{dm}{dy}p' + \frac{dm'}{dy}q'$$

$$t = n^2r' + 2nn's' + n'^2t' + \frac{dn}{dy}p' + \frac{dn'}{dy}q'.$$

53. *Required to integrate the general linear equation of the second order.*

The form is $\text{R}r + \text{S}s + \text{T}t + \text{P}p + \text{Q}q + \text{M}z = \text{N}$ (1), where $\text{R}, \text{S}, \cdots \text{N}$ are functions of (x, y).

Substituting for r, s, $\cdots$ their values in terms of r', s', $\cdots$, the result is $(\text{R}m^2 + \text{S}mn + \text{T}n^2)r' + \text{A}s' + (\text{R}m'^2 + \text{S}m'n' + \text{T}n'^2)t' + \text{B}p' + \text{C}q' + \text{M}z = \text{N}$; where A, B, C are substituted for the sake of convenience.

Now v and w being as yet undetermined, suppose them assumed such that

$$\left.\begin{aligned} \text{R}m^2 + \text{S}mn + \text{T}n^2 &= 0 \\ \text{R}m'^2 + \text{S}m'n' + \text{T}n'^2 &= 0 \end{aligned}\right\} (a);$$ then the proposed becomes $\text{A}s' + \text{B}p' + \text{C}q' + \text{M}z = \text{N}$ (2); which is to be integrated as in art. 51, and z will be found in terms of (v, w).

The two equations (a) have the same roots; let them be k and k' so that $k = \dfrac{m}{n}$, $k' = \dfrac{m'}{n'}$; then k and k' are functions which contain only x and y.

To determine v and w, we have the equations $m - kn = 0$ and $m' - k'n' = 0$, which are the same as $\frac{dv}{dx} - k\frac{dv}{dy} = 0$ and $\frac{dw}{dx} - k'\frac{dw}{dy} = 0$; which, when integrated, will give v and w in terms of arbitrary functions of (x, y). Take the simplest forms of these functions; and calculating by means of them the values of A, B, C, there will result an equation which, when integrated, will be the required primitive.

We are at liberty to take particular forms of these arbitrary functions; because in assuming for v and w, the only restriction is that the equations (a) shall be satisfied.

Ex. 1. $\frac{d^2z}{dx^2} + A\frac{d^2z}{dxdy} + B\frac{d^2z}{dy^2} + C\frac{dz}{dx} + D\frac{dz}{dy} + Ez = F.$

Let k and k' be the roots of $k^2 + Ak + B = 0$; then the equations are $\frac{dv}{dx} - k\frac{dv}{dy} = 0$, $\frac{dw}{dy} - k'\frac{dw}{dy} = 0$; whose primitives are $v = \phi(y + kx)$, $w = \psi(y + k'x)$.

Take $v = y + kx$, $w = y + k'x$; then by transformation the proposed becomes $\frac{d^2z}{dvdw} + a\frac{dz}{dv} + b\frac{dz}{dw} + cz = F'$, where

$$a = \frac{kC + D}{4B - A^2},\ b = \frac{k'C + D}{4B - A^2},\ c = \frac{E}{4B - A^2} \text{ and } F' = \frac{F}{4B - A^2}.$$

This equation, and consequently the proposed is integrable when $c = ab$.

Ex. 2. $\frac{d^2z}{dx^2} - a^2\frac{d^2z}{dy^2} + 2ab\frac{dz}{dx} + 2a^2b\frac{dz}{dy} = 0$, or

$r - a^2t + 2abp + 2a^2bq = 0.$

Here $s = 0 \therefore k^2 - a^2 = 0$, or $k = a$, $k' = -a$; and the equations for finding the new functions are $\frac{dv}{dx} - a\frac{dv}{dy} = 0$ and $\frac{dw}{dx} + a\frac{dw}{dy} = 0$; whose primitives are $v = \phi(y + ax)$ and $w = \psi(y - ax)$. (Art. 6, ex. 5.)

Take $v = y + ax$, $w = y - ax$; then, we have

$$\left.\begin{array}{l} p = ap' - aq' = a(p' - q') \\ q = p' + q' \\ r = a^2r' - 2a^2s' + a^2t' = a^2(r' - 2s' + t') \\ t = r' + 2s' + t' \end{array}\right\}, \text{ whence, by}$$

substitution, $-4a^2s' + 4a^2bp' = 0$, or $s' - bp' = 0$.

To integrate this, we have $\frac{dp'}{dw} - bp' = 0$; $\therefore$ integrating, $p' = e^{bw}\phi' v$ and $z = e^{bw}\phi v + \psi w = e^{b(y-ax)}\phi(y+ax) + \psi(y-ax)$.

Ex. 3. $\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} = \frac{2}{x}\frac{dz}{dx}$, or $r - t = \frac{2p}{x}$.

Here $k^2 - 1 = 0$ $\therefore$ $k = 1$ and $k' = -1$; and the equations are $\frac{dv}{dx} - \frac{dv}{dy} = 0$ and $\frac{dw}{dx} + \frac{dw}{dy}$; and taking the primitives $v = y + x$ and $w = y - x$, we have $p = p' - q'$, $r = r' - 2s' + t'$, and $t = r' + 2s' + t'$; $\therefore$ by substitution, $-4s' = \frac{2}{x}(p' - q')$, or $s' + \frac{1}{v-w}(p' - q') = 0$; whose primitive is $z = \varphi v + \psi w - \frac{v-w}{2}\left\{\phi' v - \psi' w\right\}$ . . . .

$$= \varphi(y+x) + \psi(y-x) - x\left\{\phi'(y+x) - \psi'(y-x)\right\}.$$

Ex. 4. $\frac{d^2z}{dx^2} - \frac{y^2}{x^2}\frac{d^2z}{dy^2} + \frac{1}{y}\frac{dz}{dx} - \frac{1}{x}\frac{dz}{dy} = 0$.

Here $k = \frac{y}{x}$, $k' = -\frac{y}{x}$ and the equations are $\frac{dv}{dx} - \frac{y}{x}\frac{dv}{dy} = 0$ and $\frac{dw}{dx} + \frac{y}{x}\frac{dw}{dx} = 0$; whose primitives are $v = \phi.xy$ and $w = \frac{y}{x}$.

Take $v = xy$, $w = \frac{y}{x}$; then by means of the same substitution as before, the proposed becomes $\frac{-4y^2s'}{x^2} - \left\{\frac{2}{x^2} - \frac{2y}{x^3}\right\}q' = 0$, or $2w^2s' + \frac{w}{v}(1-w)q' = 0$, or $2w\frac{dq'}{dv} + \frac{1-w}{v}q' = 0$ $\therefore$ $2w\frac{dq'}{q'} + (1-w)\frac{dv}{v} = 0$; and in-

tegrating this as a total equation between q', v and w, there results $q'^{2w}.v^{1-w} = \psi' w$; or q', which $= \frac{dz}{dw}$, $= v^{\frac{w-1}{2w}}\psi w$ $\therefore$ $z = \int v^{\frac{w-1}{2w}}\psi w dw + \phi v = \phi v + v^{\frac{1}{2}}\int v^{-\frac{1}{2w}} d.\psi w$; where $v = xy$ and $w = \frac{y}{x}$.

There are no general methods of integrating equations in which the fluxional coefficients are not linear. The two following cases have been integrated by Legendre.

54. *Required to integrate* $\mathrm{R}\frac{d^2z}{dx^2} + \mathrm{S}\frac{d^2z}{dxdy} + \mathrm{T}\frac{d^2z}{dy^2} = 0$, *where* R, S *and* T *are functions which contain only* $\frac{dz}{dx}$ *and* $\frac{dz}{dy}$.

The following method consists in transforming the proposed into $\mathrm{R}r' - \mathrm{S}s' + \mathrm{T}t' = 0$, where r', s' and t' are the fluxional coefficients of $v = f(p, q)$.

Since $dz = d(px + qy) - (xdp + ydq)$, we have $xdp + ydq$ an exact fluxion. Let $xdp + ydq = dv$.

In this equation p and q may be considered as the independent variables instead of x and y; and the values of r, s and t may be deduced on this hypothesis.

First, to find r, we have (ex hyp.) $\frac{dv}{dp} = x$, and consequently $dx = \frac{d^2v}{dp^2}dp + \frac{d^2v}{dpdq}dq$ (1). Also, in calculating the value of r, y or $\frac{dv}{dq}$ is supposed constant; wherefore we likewise have $0 = \frac{d^2v}{dqdp}dp + \frac{d^2v}{dq^2}dq$; and eliminating dq from (1), there results $dx = \left\{\frac{d^2v}{dp^2} - \frac{d^2v^2}{dp^2dq^2} \div \frac{d^2v}{dq^2}\right\} dp = \frac{mdp}{\frac{d^2v}{dq^2}}$, if $m = \frac{d^2v}{dp^2}\frac{d^2v}{dq^2} - \frac{d^2v^2}{dp^2dq^2}$.

But $r = \frac{dp}{dx}$; wherefore $r = \frac{1}{m}\frac{d^2v}{dq^2}$.

Similarly it may be shown that $s = -\frac{1}{m}\frac{d^2v}{dpdq}$; and $t = \frac{1}{m}\frac{d^2v}{dp^2}$, and by substitution the proposed becomes $Rr' - Ss' + Tt' = 0$, where R, S and T are functions of the independent variables; which therefore is to be integrated by Laplace's method. (Art. 53.)

Having thus obtained v in terms of (p, q), the required primitive is to be found either by integrating $z = px + qy - v$, or by eliminating p and q by means of the values of x and y.

55. *Required to integrate* $R\frac{d^2z}{dx^2} + S\frac{d^2z}{dxdy} + T\frac{dz^2}{dy^2} = 0$, *which is not linear, and in which* R, S *and* T *are functions which contain only* $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dxdy}$ *and* $\frac{dz^2}{dy^2}$.

Since $r = f(s, t)$, let $dr = Pds + Qdt$, where P and Q are known functions of (s, t).

Since $\left.\begin{aligned} dp &= rdx + sdy = d(rx + sy) - (xdr + yds) \\ dq &= sdx + tdy = d(sx + ty) - (xds + ydt) \end{aligned}\right\}$, we have $xdr + yds$ and $xds + ydt$ both exact fluxions; or x and y may be considered as functions of (r, s, t), and consequently of (s, t).

Let $xds + ydt = dv$; then $x = \frac{dv}{ds}$ and $y = \frac{dv}{dt}$. Also $xdr + yds$ becomes $x(Pds + Qdt) + yds$ or $(xP + y)ds + xQdt$, which being an exact fluxion, we have $\frac{d(xP+y)}{dt} = \frac{d.xQ}{ds}$; whence, since $\frac{dP}{dt} = \frac{dQ}{ds}$, $P\frac{dx}{dt} + \frac{dy}{dt} = Q\frac{dx}{ds}$. But $\frac{dx}{dt} = \frac{d^2v}{dsdt}$; $\frac{dy}{dt} = \frac{d^2v}{dt^2}$ and $\frac{dx}{ds} = \frac{d^2v}{ds^2}$; wherefore $\frac{d^2v}{dt^2} + P\frac{d^2v}{dsdt} - Q\frac{d^2v}{ds^2} = 0$; which, when integrated by Laplace's method, will give v in terms of (s, t).

Having found v, we may obtain x, y, z each in terms of (s, t).

For $x = \frac{dv}{ds}$ (1); $y = \frac{dv}{dt}$ (2). Also to obtain z; we have $p = rx + sy - \int(xdr + yds)$, in which $xdr + yds$ being an exact fluxion, and expressed in terms of (s, t), it may be integrated, and p be found in terms of (s, t). And $q = sx + ty - v$; wherefore z, which $= px + qy - \int(xdp + ydq)$ may be also found in terms of (s, t); and eliminating s and t by means of these *three* equations, there will result the required primitive.

We shall give but one more instance of a general form; and then terminate this part of the subject with some examples that are to be integrated by particular methods.

$$56.\ \frac{d^nz}{dx^n} + \text{A}\frac{d^nz}{dx^{n-1}dy} + \text{B}\frac{d^nz}{dx^{n-2}dy^2} \cdots + \text{E}\frac{d^nz}{dy^n} = 0.$$

It appears by differentiation that the proposed is satisfied by $z = \phi(y + kx)$ where k is a root of $k^n + \text{A}k^{n-1} \cdots + \text{E} = 0$. Let $k', k'' \cdots k^{(n)}$ be the n roots of this equation; then it may be shown as in ch. 6, 20, that the primitive is $z = \phi'(y + k'x) + \phi''(y + k''x) \cdots + \phi^{(n)}(y + k^{(n)}x)$, which contains n arbitrary functions.

If $k' = k''$, $z = \phi'(y + k'x) + x\phi''(y + k'x) \cdots + \phi^{(n)}(y + k^{(n)}x)$.

57. EXAMPLES.

Ex. 1. $\frac{d^3z}{dx^3} - (2a + b)\frac{d^3z}{dx^2dy} + (2ab + a^2)\frac{d^3z}{dxdy^2} - a^2b\frac{d^3z}{dy^3} = 0$

$\therefore z = \phi(y + ax) + x\psi(y + ax) + \chi(y + bx)$.

Ex. 2. $\frac{d^2z}{dx^2} = a^2z$.

Substitute $\frac{dz}{dx} = kz \therefore \frac{d^2z}{dx^2} = k\frac{dz}{dx} = k^2z$, which $\therefore = a^2z$,

$\therefore k = 1, k' = -1$ and $z = e^{ax}\phi y + e^{-ax}\psi y$.

Ex. 3. $\frac{d^3z}{dx^3} = a^3z$.

Substitute $\frac{dz}{dx} = kz \therefore \frac{d^3z}{dx^3} = k^3z = a^3z \therefore k = a, k' = ma$

and $k'' = na$, where 1, m and n are the cube roots of unity;

and $z = e^{ax}\phi y + e^{-\frac{ax}{2}}\cos.\frac{ax\sqrt{3}}{2}\chi y + e^{\frac{ax}{2}}\sin.\frac{ax\sqrt{3}}{2}\psi y$.

Ex. 4. $(a+2b)z-(2a+3b)\frac{dz}{dx}+c\frac{dz}{dy}$ . . . . .

$+a\frac{d^2z}{dx^2}-2c\frac{d^2z}{dxdy}+b\frac{d^3z}{dx^3}+c\frac{d^3z}{dx^2dy}=0.$

Substitute $z=e^xv$; then the proposed becomes

$$0=(a+3b)\frac{d^2v}{dx^2}+b\frac{d^3v}{dx^3}+c\frac{d^3v}{dx^2dy}.$$

Substitute $w=\frac{d^2v}{dx^2}$; and this becomes

$$0=(a+3b)w+b\frac{dw}{dx}+c\frac{dw}{dy}.$$

If this be integrated, there will result

$$z=e^x\left\{e^{-\frac{my}{a}}\varphi(cx-by)+x\chi y+\psi y\right\}, \text{ where } m=a+3b.$$

Ex. 5. $\frac{d^2z}{dx^2}+b\frac{d^2z}{dxdy}-2a\frac{dz}{dx}-ab\frac{dz}{dy}+a^2z=0.$

Substitute $z=e^{ax}v$; then

$$\frac{d^2v}{dx^2}+b\frac{d^2v}{dxdy}=0 \text{ and } z=e^{ax}\left\{\varphi(y-ax)+\psi y\right\}.$$

Ex. 6. $\frac{d^4z}{dx^4}=a^2\frac{d^2z}{dx^2}.$

It appears by differentiation that the proposed is satisfied by $\frac{d^2z}{dy^2}=k\frac{dz}{dx}$, where $k^2=a^2$ or $k=\pm a$; which divides it into

$$\left.\begin{aligned}\frac{d^2z}{dy^2}-a\frac{dz}{dx}&=0\\ \frac{d^2z}{dy^2}+a\frac{dz}{dx}&=0\end{aligned}\right\}. \text{ Let } \left.\begin{aligned}z&=\text{P}\\ z&=\text{Q}\end{aligned}\right\} \text{ be the primitives of these:}$$

then the required primitive is $z=\text{P}+\text{Q}$, which contains four arbitrary constants.

Ex. 7. $\frac{d^4z}{dy^4}=a^2\frac{d^2z}{dx^2}+2ab\frac{dz}{dx}+b^2z.$

This is reducible as in the last example to the two,

$$\frac{d^2z}{dy^2}=\pm a\frac{dz}{dx}+bz.$$

For more examples of similar reductions, vid. Euler's Int. Calc. tome iii. ch. 2, p. 359.

Ex. 8. $\frac{d^2z}{dx^2}\frac{d^2z}{dy^2} = \frac{d^2z^2}{dx^2dy^2}$, or $rt = s^2$; which is the equation of developable surfaces.

Substitute $\frac{r}{s} = \frac{s}{t} = v \therefore r = vs, s = vt \therefore$

$dp = rdx + sdy = v(sdx + tdy) = vdq$, or $p = \varphi q$, which is therefore a first primitive.

The canonical primitive obtained as in art. 25 is

$$\left.\begin{aligned} y - \phi a &= (z - a)\varphi' a \\ x - \psi a &= (z - a)\psi' a \end{aligned}\right\};$$ where a is to be eliminated, having assigned a form to ϕ.

57. We shall omit the cases in which z is a function of three or more variables; though they frequently occur. The truth is, that the student must consider this chapter as containing a very imperfect account of the doctrine of partial equations. It would require a distinct treatise to discuss fully this branch of the subject. When the equations are of the first order, they are always integrable; or at least they may be considered to be so; but of those of the second order it cannot be predicated that to each equation there belongs a fluent containing two arbitrary functions. The student will find numerous instances of the integration of particular forms in the works of Euler and Monge and in Poisson's Mémoire on Sound, and various other Memoires in the volumes of the Ecole Polytechnique.

58. *On the construction of equations; and the determination of the arbitrary constants.*

The arbitrary functions in partial equations are to be determined as the arbitrary constants are in total equations from the conditions of the question. If, after having used all the conditions, any remain, the problem is in some respects indeterminate; but the result will show in what sense and to what extent it is so.

59. *Required to construct the surface to which belongs the equation* $\frac{dz}{dx} = a$, *where* z *is a function of* (x, y).

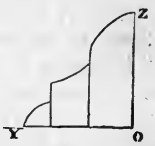
The primitive is $z = ax + \phi y$ (α).

Adopting the same notation as before, $dz = pdx + qdy$; and, referring to the diagram of vol. i. ch. 9. 17, it results

from the notation that $dz = pdx$ is an equation which belongs to the curve TPV; and $dz = qdy$, to the curve SPR; for y is constant in the one case, and x in the other.

Here $\frac{dz}{dx} = a$ or $z : x$ is a constant ratio; and consequently TPV is a right line. The same results from (α); for, substituting $c = \varphi y$, it becomes $z = ax + c$, which is a right line inclined to the plane XOY at tan.^{-1}a.

Suppose $x = 0$; then (α) becomes $z = \phi y = c$. Assigning a particular value to c, this represents only one point in the curve ZY; but in general $z = \phi y$ represents a curve ZY, which is as arbitrary as possible. It may be *discontinuous*, *i. e.* consisting of two or more different curves; or it may be *discontiguous*, in which case the parts are not only different curves; but at their junction, they have different ordinates to the same abscissa as in the annexed diagram.



Suppose the discontinuous curve to consist of an indefinite number of small arcs; then, in the limit, it may represent an *irregular* curve, *i. e.* one which is described at random not subject to any law.

Describe then (liberâ manu) any curve whatever in the plane ZOY; then a right line moving along this curve parallel to itself and to the plane ZOX and inclined to XOY at tan.^{-1}a will describe a surface which belongs to $\frac{dz}{dx} = a$.

60. *Required to construct the surface belonging to* $\frac{dz}{dx} = \text{F}x$.

The primitive is $z = \int \text{F}x.dx + \phi y$. The surface is constructed in the same manner as in the last article, except that the generating line is now a curve, whose equation is $z = \int \text{F}x.dx + c$, and whose nature depends upon the form of F.

The problem is as before indeterminate, *i. e.* the curve ZY may be described at random; but, when once described, the surface is traced by the *same* curve $z = \int \text{F}x.dx + c$ moving along it parallel to itself and to the plane ZOX.

61. *Required to construct the surface belonging to* $\frac{dz}{dx} = \text{P}$, *where* $\text{P} = f(x, y)$.

The generating line TPV is, as before, a curve whose equation is $z = \int \text{P}dx + \phi y$. Let y become successively y', y'', $y''' \cdots$; and let the corresponding values of P be P', P'', P''' $\cdots$; then the generating curves are $z = \int \text{P}'dx + \phi y'$; $z = \int \text{P}''dx + \phi y''$; ... These are all of the same *kind*, differing only in magnitude and position in consequence of the parameters, which contain y, varying.

When $x = 0$, let the equation become $z = \text{Y} + \phi y$; which therefore is the equation of the arbitrary curve ZY. Having described this, the required surface will be traced by a curve whose equation is $z = \int \text{P}dx + c$ moving along it parallel to itself and to the plane ZOX.

The same construction obtains if $\text{P} = f(x, y, z)$.

These constructions also follow from the consideration, that when the equation is $\frac{dz}{dx} = \text{F}(x, y, z)$, since $\frac{dz}{dy}$ does not enter, it cannot be determined from the equation, and consequently the transverse curve is arbitrary.

If both $\frac{dz}{dx}$ and $\frac{dz}{dy}$ enter into the equation, the problem is still indeterminate; for in partial equations $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are independent of each other; it is this which constitutes the characteristick difference between partial and total equations. In the latter the fluxional coefficients are not wholly independent, for they are connected by the condition $\frac{dp}{dy} = \frac{dq}{dx}$.

62. *Required to construct the surface* $\text{P}\frac{dz}{dx} + \text{Q}\frac{dz}{dy} = 0$, *where* P *and* Q *are functions of* (x, y).

Let $t(\text{Q}dx - \text{P}dy)$ be an exact fluxion $= ds$; then the equations (α) and (β) of art. 9 give $s = a$, $z =$ a constant $= b$; and the primitive is $z = \phi s$. (art. 9. cor.)

The equation $s = a$ between x and y is of a known form. Describe the curve which it represents in the plane XY, and describe upon this curve a cylinder whose axis is parallel to Z or perpendicular to XY; then since $s = a$ belongs both to

the required and to the cylindrical surface, and that at their common section we have $z = b$, one property is that this common section is parallel to XY, or perpendicular both to XZ and to YZ.

Hence to construct the required surface; in the equation $s = a$ give to a all possible values from 0 to ∞, which will not alter the nature of the curve. Describe upon the curve a cylindrical surface perpendicular to XY. Let a plane move at random along Z parallel to XY; then the intersections of the plane and of the cylinder will generate the required surface; which may or may not follow the law of continuity.

63. *Required to construct the general equation,*

$$P\frac{dz}{dx} + Q\frac{dz}{dy} = R.$$

Let the primitive be $s = \phi s'$ where $s = a$, $s' = b$; s and s' being functions of (x, y, z); and a and b the constants introduced by the integration.

By eliminating z and y, two equations may be obtained of the form $F(x, y, a, b) = 0 = u$, $f(x, z, a, b) = 0 = v$; or $u = 0$, $v = 0$ may be considered as the projections of any arc of the required surface on the planes XY, XZ respectively. Hence to construct the surface; describe upon these planes the known curves $u = 0$, $v = 0$, and also perpendicular cylindrical surfaces; then their intersection is an arc of the required surface; and if the curves' parameters a and b be supposed to vary as in the last article, this arc will trace the required surface, which may be regular or irregular.

It appears then that the problem of determining the arbitrary function, of which we shall soon give an analytical solution, is nothing more than the making a surface pass through a curve which is drawn arbitrarily in fixed space.

Similar constructions obtain where the equation is of the second order; but in this case, since the primitive contains two arbitrary functions, the problem of determining them amounts to the making a surface pass through two arbitrary curves. We shall give only two instances, and those of the simplest form.

64. *Required to construct* $\dfrac{d^2z}{dxdy} = 0.$

The primitive is $z = \phi x + \psi y$; whence, if $y = 0$, $z = \phi x$; and if $x = 0$, $z = \psi y$.

Describe then the arbitrary curves $z = \phi x$, $z = \psi y$ in the planes XZ, YZ respectively. Let z' be the ordinate corresponding to x' in $z = \phi x$, and z'' the ordinate corresponding to y' in $z = \psi y$; then at the point (x', y') of the plane XY erect a perpendicular $= z' + z''$; which evidently traces the required surface.

65. *Required to construct* $\dfrac{d^2z}{dx^2} = 0$.

The primitive is $z = x\psi y + \psi y$; whence, if $x = 0$, $z = \psi y$.

Describe in the plane ZOY the arbitrary curve $z = \psi y$; let it be ZY. Through any point T in ZY draw a plane TMV perpendicular to XOY; this will intersect the required surface in a right line; for, y being constant, $z = ax + b$, where $b = \psi y$.

It does not appear so immediately as in the last article that we have the means of determining the position of this line from the other function ϕ; but if we suppose $x = 1$, we shall have $z = \phi y + \psi y$. Draw then a plane SNR parallel to ZOY and at a distance from it $= 1$; describe in it the arbitrary curve $z = \phi y + \psi y$, differing from $z = \psi y$ only in this, that the ordinate instead of being $z = \psi y$ is $z = \varphi y + \psi y$; then the right line joining the corresponding points of the two curves shall generate the required surface; which may be regular or irregular.

66. *On the determination of the arbitrary functions.*

The arbitrary functions are to be determined from the conditions of the question; and it is obvious that we must have as many conditions as there are functions to be determined. The conditions consist in the function z taking a known form, when certain relations are assigned to x and y. We shall illustrate this by only two instances; the one, of the first; and the other, of the second order.

67. *Required to determine the arbitrary function in* $1 = s\phi s'$, *where* s *and* s' *are certain functions of* (x, y, z).

Suppose the condition is that $F(x, y, z) = 0$ when $f(x, y, z) = 0$, where F and f denote given forms. To determine ϕ; we have the three equations $F(x, y, z) = 0$; $f(x, y, z) = 0$; and $s' =$ function of (x, y, z). By means of these, each of the variables x, y, z may be expressed in

terms of s'. Substituting these values in s, let it become s'; then the proposed equation becomes $1 = s'\phi s'$ or $\phi s' = \frac{1}{s'}$; which determines ϕ, and a definite form may be thus given to $1 = s\phi s'$.

68. *Required to determine the arbitrary function in* $1 = s\phi s'' + s'\psi s''$, *where* s, s', s'' *are functions of* x, y, z.

Here there are two functions to be determined; and we must in consequence have two conditions. Suppose that $\mathrm{F}(x, y, z) = 0$ gives $f(x, y, z) = 0$; and that $\mathrm{F}'(x, y, z) = 0$ gives $f'(x, y, z) = 0$.

By means of $\mathrm{F}(x, y, z) = 0, f(x, y, z) = 0$ and $s'' = \text{funct.}(x, y, z)$, the variables x, y, z may be expressed in terms of s''.

Substituting these values in s and s', let them become s'' and σ''; then we have $1 = s''\phi s'' + \sigma''\psi s''$ (1). Also by means of $\mathrm{F}'(x, y, z) = 0, f'(x, y, z) = 0$ and $s'' = \text{function}$ of (x, y, z), the variables may be expressed under another form in terms of s''. Let s and s' in consequence become s' and σ'; then we have $1 = s'\varphi s'' + \sigma'\psi s''$ (2). By means of (1) and (2) eliminating either function, the other may be determined, and thus a definite form may be given to the proposed equation.

The same method is evidently applicable to any equation which is under the form $1 = s\phi s'^n + s'\phi s'^n + s''\phi s'^n + \cdots$ provided that we have n conditions.

If the given equation contains the fluxional coefficients, or if it be not under the above form, we cannot determine the functions by means of the fluxional calculus; recourse must be had to a different branch of analysis. Vid. Mr. Herschell's note (P) in the Cambridge edition of Lacroix, p. 701.

69. EXAMPLES.

PROB. 1. Required the equation of a cylindrical surface whose base is an ellipse in the plane XY; and whose axis is parallel to a given line.

The characteristick property of a cylinder is that its tangent plane is always parallel to the axis.

Let the axis, which may be drawn through the origin of the co-ordinates, be $\left.\begin{matrix} x = mz \\ y = nz \end{matrix}\right\}$. The tangent plane of the cylinder is $z' - z = p(x' - x) + q(y' - y)$ (vol. i. ch. 8.

19); and this is to be always parallel to $\left.\begin{matrix} x = mz \\ y = nz \end{matrix}\right\}$; wherefore (vol. i. ch. 7. 68. cor.) $mp + nq = 1$ (1); which is therefore the equation of condition that the surface may be cylindrical.

The canonical primitive of (1) is $x - mz = \phi(y - nz)$ (2), which belongs to all cylindrical surfaces; and to determine ϕ in this case, let the cylinder's base be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $x = \frac{a}{b}\sqrt{a^2 - y^2}$; but when $z = 0$, we have from (2), $x = \phi y$; whence $\phi y = \frac{a}{b}\sqrt{a^2 - y^2}$; and the required equation is $x - mz = \frac{a}{b}\sqrt{a^2 - (y - nz)^2}$.

The canonical primitive may be also obtained in the following manner.

The generating line is always parallel to the axis; let it be $\left.\begin{matrix} x = mz + \alpha \\ y = nz + \beta \end{matrix}\right\}$; then $\left.\begin{matrix} \alpha = x - mz \\ \beta = y - nz \end{matrix}\right\}$.

Now, if the position of the generating line be changed, m and n remain unaltered, but α and β vary; *i. e.* α and β are quantities such that α varies when β varies; and is constant when β is constant; and consequently $\alpha = \phi\beta$, or $x - mz = \phi(y - nz)$.

Prob. 2. Required the equation of conical surfaces.

Def. A *cone* is generated by a right line, which always passes through a given point called the *apex*, moving along a curve called the directrix.

A characteristick property of a cone, which is independent of the nature of the directrix, is that a tangent plane always passes through the apex.

Let (a, b, c) be the apex. The tangent plane is $z - z' = (x - x')p + (y - y')q$. But this passes through (a, b, c); wherefore the required equation of condition is $z - c = (x - a)p + (y - b)q$ (1).

The primitive of this is $\frac{y-b}{z-c} = \phi\frac{x-a}{z-c}$: and to determine ϕ, let the directrix be $\left.\begin{matrix} F(x, z) = 0 \\ f(y, z) = 0 \end{matrix}\right\}$.

By means of these two equations combined with

$\frac{x-a}{z-c} = \alpha$ and $\frac{y-b}{z-c} = \varphi\alpha$, we can eliminate x, y and z, and thus determine the form of φ; let the result be $F(\alpha, \phi\alpha) = 0$, where F represents a known form; then the required equation is $F\left\{\frac{x-a}{z-c}, \frac{y-b}{z-c}\right\} = 0$.

The canonical primitive, as in the first problem, may be found independently of the calculus.

PROB. 3. Required to find from the same data the conical surface which touches a given curved surface.

The cone touches the surface in some curve whose equations are to be found from the data; and this curve is the cone's directrix.

Now at every point of the directrix, the cone and the given surface have common co-ordinates; and also the values of p and q are the same, since they have the same tangent plane; find then the values of p and q from the given equation; substitute them in the cone's equation $z - c = (x - a)p + (y - b)q$ (1), and let the result be $F'(x, y, z) = 0$; in which x, y and z belong to the common section of the cone and surface, *i. e.* to the directrix. Let $F'(x, y, z) = 0$, when projected on the planes XZ and YZ, give $\left.\begin{matrix} F(x, z) = 0 \\ f(y, z) = 0 \end{matrix}\right\}$; by means of which the conical surface may be found as in the last problem.

If an eye be situated in the apex, the directrix separates the visible from the invisible part of the surface.

EX. Let the surface be of the second degree, whose general equation is

$$\left.\begin{matrix} Ax^2 + A'y^2 + A''z^2 \\ + 2Bxy + 2B'xz + 2B''yz \\ + Cx + C'y + C''z \end{matrix}\right\} = D,$$

$$\therefore p = -\frac{Ax + By + B'z + C}{B'x + B''y + A''z + C''} \text{ and } q = -\frac{Bx + A'y + B''z + C'}{B'x + B''y + A''z + C''};$$

and if these be substituted in (1), and the original equation be added, there will result an equation to a plane surface, which shows that the points of the directrix lie in the same plane, and that it is not a curve of double curvature.

If the surface be a sphere, it is manifest that the directrix is a small circle of the sphere at right angles to the line joining its centre and the apex of the cone.

It may be shown in the same manner, that if the curved surface is of the mth degree, the equation $f(x, y, z) = 0$ is

of $(m-1)$ dimensions, or the directrix, when a cone is the touching surface, is a curve of double curvature drawn upon a surface of the $(m-1)$th degree.

PROB. 4. Required the equation of surfaces of revolution.

Def. The plane drawn through the axis of revolution and any point of the surface is called a *meridian* plane.

A characteristick property of a surface of revolution, which is independent of the nature of the generating curve, is that a tangent plane is at all points perpendicular to the meridian plane.

Let the axis be $\left.\begin{matrix} x = az + \alpha \\ y = bz + \beta \end{matrix}\right\}$; then the meridian plane is $z - z' = \frac{1}{a}(x - x') + \frac{1}{b}(y - y')$ (vol. i. ch. 7. 75);

and the tangent plane is

$z - z' = p(x - x') + q(y - y')$, where x, y, z are the co-ordinates of any point in the required surface *. But these are normal planes; wherefore $\frac{p}{a} + \frac{q}{b} + 1 = 0$ (1) (vol. i. ch. 7. 70, cor. 1); which is therefore the equation of condition in order that the surface may be one of revolution round an axis which is parallel to the line $\left.\begin{matrix} x = az \\ y = bz \end{matrix}\right\}$.

Next to find an equation contingent upon the *position* of the axis; we have the equations of the normal $\left.\begin{matrix} x - x' + p(z - z') = 0 \\ y - y' + q(z - z') = 0 \end{matrix}\right\}$; and, since the normal passes through the axis, these equations are simultaneous with $\left.\begin{matrix} x' = az' + \alpha \\ y' = bz' + \beta \end{matrix}\right\}$; whence, eliminating x', y', z', there results $(y - \beta - bz)p - (x - \alpha - az)q = b(x - \alpha) - a(y - \beta)$ (1); which is the equation of condition that the surface may be generated by a line revolving round the axis $\left.\begin{matrix} x' = az' + \alpha \\ y' = bz' + \beta \end{matrix}\right\}$.

The primitive of (1) is

$(x - \alpha)^2 + (y - \beta)^2 + z^2 = \phi(ax + by + z)$ (2). If the

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* It is here supposed that the tangent plane meets the axis in a point (x', y', z').

axis passes through the origin, this becomes $x^2 + y^2 + z^2 = \phi(ax + by + z)$. If it coincides with z, we have $a=0$ and $b=0$; $x^2 + y^2 + z^2 = \phi z$, or $x^2+y^2=\psi z$; which is therefore the equation of all surfaces of revolution whose axes coincide with z.

To determine ψ; let the generating curve be an inverted parabola, whose axis coincides with z; then, when the curve is in the plane zy, $y^2 = \psi z$; but (ex hyp.)$y^2 = mz$; whence $\psi z = mz$, and the required equation is $x^2 + y^2 = mz$.

Prob. 5. Required the curved surface in which all the normals meet a given plane in a given right line.

Take xz for the given plane; and let the given line pass through the origin, its equation being $z = ax$.

The equations of the normal are

$$\left.\begin{aligned} x - x' + p(z - z') &= 0 \ (1) \\ y - y' + q(z - z') &= 0 \ (2) \end{aligned}\right\},$$

where (x, y, z) is any point in the surface, and (x', y', z') the corresponding point in the given right line.

Suppose $y' = 0$; then (2) becomes $y + q(z - z') = 0$, or

$z - z' = -\dfrac{y}{q}$; and by elimination from (1) we have

$x - x' = \dfrac{py}{q}$, and consequently $z' = \dfrac{y}{q} + z$ and

$x' = x - \dfrac{py}{q}$. But (ex hyp.) $z' = ax'$; whence

$\dfrac{y}{q} + z = ax - \dfrac{apy}{q}$, or $ayp + (z - ax)q = -y$.

This may be integrated as in art. 9; but if we change the axes so that the given line may coincide with z, we have $a = \infty$ and the equation becomes $yp - xq = 0$, whose primitive is $z = \phi(x^2 + y^2)$ (Art. 6, ex. 3.)

This result shows that the problem is to a certain degree indeterminate, as we have not the means of determining ϕ; but it also shows that the proposed property belongs alone to surfaces of revolution *.

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* These problems properly belong to our chapter on curved surfaces; but as we shall be compelled to omit this chapter for want of space, they are inserted here to illustrate the theory of arbitrary functions.

70. MISCELLANEOUS PRAXIS.

1. $\sqrt{a^2-y^2-x^2}\frac{dz}{dx}=a$; $\therefore z=a\sin.^{-1}\frac{x}{\sqrt{a^2-y^2}}+\phi y$.

2. $x\frac{dz}{dx}=nz \therefore z=x^n\phi y$.

3. $xy\frac{dz}{dy}+az=0 \therefore y^az^x=\varphi x$.

4. $(x^2+y^2)\frac{dz}{dx}=y^2+z^2 \therefore z=\frac{y^2\phi y+x}{1-x\phi y}$.

5. $x\frac{dz}{dx}+y\frac{dz}{dy}=n\sqrt{x^2+y^2} \therefore z=n\sqrt{x^2+y^2}+\varphi\frac{y}{x}$.

6. $x^2\frac{dz}{dx}+y\frac{dz}{dy}=y^2 \therefore z=\frac{y^2}{2}+\psi.ye^{\frac{1}{x}}$.

7. $\frac{dz}{dy}=\mathrm{X}\frac{dz}{dx}+\mathrm{X}' \therefore z=-\int\frac{\mathrm{X}'dx}{\mathrm{X}}+\phi\left\{y+\int\frac{dx}{\mathrm{X}}\right\}$.

8. $\mathrm{X}\frac{dz}{dx}+\mathrm{Y}\frac{dz}{dy}=\mathrm{Y}' \therefore z=\int\frac{\mathrm{Y}'}{\mathrm{Y}}dy+\phi\left\{\int\frac{dy}{\mathrm{Y}}-\int\frac{dx}{\mathrm{X}}\right\}$.

9. $x\frac{dz}{dx}+y\frac{dz}{dy}=\frac{xy}{z} \therefore z^2-xy=\phi\frac{y}{x}$.

10. $xy\frac{dz}{dx}+x^2\frac{dz}{dy}=yz \therefore z=x\phi(y^2-x^2)$.

11. $x\frac{dz}{dx}-y\frac{dz}{dy}=\frac{y^2}{z} \therefore z^2+y^2=\varphi.xy$.

12. $x\frac{dz}{dx}+z\frac{dz}{dy}+y=0 \therefore xe^{\sin.^{-1}\frac{z}{\sqrt{x^2+y^2}}}=\phi(x^2+y^2)$.

13. $x\frac{dz}{dx}+z\frac{dz}{dy}=y \therefore z+y=x\varphi.x(2-y)$.

14. $x^2\frac{dz}{dx}+y^2\frac{dz}{dy}=z^2 \therefore \frac{1}{z}-\frac{1}{x}=\varphi\left\{\frac{1}{x}-\frac{1}{y}\right\}$.

15. $x^2\frac{dz}{dx}+y^2\frac{dz}{dy}=yz^2 \therefore e^{-\frac{1}{z}}=y\varphi\frac{y-x}{xy}$.

16. $\mathrm{X}\frac{dz}{dx}+\mathrm{Y}\frac{dz}{dy}=\mathrm{Z} \therefore \int\frac{dz}{\mathrm{Z}}-\int\frac{dx}{\mathrm{X}}=\varphi\left\{\int\frac{dy}{\mathrm{Y}}-\int\frac{dx}{\mathrm{X}}\right\}$.

17. $z=a+y+\frac{bx^2}{y^2} \therefore z=y+\varphi\frac{y}{x}$.

18. $z = x\frac{dz}{dx} + y\frac{dz}{dy} + n\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}$ $\therefore$ . . . .

$z = \sqrt{n^2 - x^2 - y^2}$ is a singular solution.

19. $z = x\frac{dz}{dx} + y\frac{dz}{dy} + 3a\left(\frac{dz}{dx}\frac{dz}{dy}\right)^{\frac{1}{3}}$ $\therefore$ $xyz = a^3$ is a singular solution.

20. $\frac{dz}{dy} = z\frac{dz^2}{dy^2}$ $\therefore$ $z^2 = 2ax + 2a^2y + b$.

21. $\frac{dz^2}{dx^2} + \frac{dz^2}{dy^2} = 1$ $\therefore$ $z + a = \sqrt{x^2 + y^2}$.

22. $x^2\frac{dz^2}{dx^2} = y\frac{dz^2}{dy^2}$ $\therefore$ $z = \psi(lx \pm 2\sqrt{y})$.

23. $\frac{dz}{dx}\frac{dz}{dy} = mx + n\frac{dz^2}{dx^2}$ $\therefore$ . . . . . .

$z = x\sqrt{2my - a} + \frac{n}{3m}(2my - a)^{\frac{3}{2}} + b$.

24. $x\frac{dz}{dx} + y\frac{dz}{dy} = n\frac{dz}{dx}\frac{dz}{dy}$ $\therefore$ $z = \frac{x}{n}(y + \sqrt{y^2 + a^2}) + b$.

25. $\frac{d^2z}{dxdy} = xy$ $\therefore$ $z = \varphi x + \psi y + \frac{x^2y^2}{4}$.

26. $x\frac{d^2z}{dx^2} = (n-1)\frac{dz}{dx}$ $\therefore$ $nz = x^n\varphi y + \psi y$.

27. $\frac{d^2z}{dx^2} - \frac{2x}{x+y}\frac{dz}{dx} = \frac{x}{ay}$ $\therefore$ . . . . . .

$z = \left(\frac{x^3}{3} + xy^2\right)\varphi y + \psi y + \frac{x^3+3xy^2}{6ay}l(x^2 + y^2)\cdots$

$-\frac{1}{3ay}\left\{\frac{x^3}{3} + 2xy^2 - 2y^2\tan.^{-1}\frac{x}{y}\right\}$.

28. $\frac{d^2z}{dxdy} + \frac{a}{y}\frac{dz}{dx} + \frac{b}{x}\frac{dz}{dy} + \frac{ab}{xy}z = \mathrm{v}$ $\therefore$ . . . . .

$z = y^{-a}\phi x + x^{-b}\psi y + y^{-a}x^{-b}\int y^a dy \int \mathrm{v}x^b dx$.

29. $\frac{d^2z}{dxdy} + \frac{a+2}{x+y}\frac{dz}{dx} + \frac{b}{x+y}\frac{dz}{dy} + \frac{ab+2b-a-2}{(x+y)^2}z = 0$

$\therefore z = (x+y)^{-a-x}\{\varphi x + \int (x+y)^{a-b+2}\psi y.dy\}$.

30. $xy\frac{d^2z}{dxdy} - nx\frac{dz}{dx} - my\frac{dz}{dy} + mnz = 0 \therefore$ . . . . .

$z = x^m\varphi y + y^n\psi x.$

31. $\frac{d^2z}{dxdy} + \frac{1}{x+y}\left\{\frac{dz}{dx} + \frac{dz}{dy}\right\} = 0 \therefore (x+y)z = \phi x + \psi y.$

32. $\frac{d^2z}{dxdy} + \frac{1}{x-y}\left\{\frac{dz}{dx} + \frac{dz}{dy}\right\} \therefore$ . . . . . .

$z = (x-y)\left\{\phi x + \int\frac{\psi y\,dy}{(x-y)^2}\right\}.$

33. $\frac{d^2z}{dxdy} + \frac{1}{x-y}\left\{\frac{dz}{dx} - \frac{dz}{dy}\right\} = 0 \therefore$ . . . . . .

$z = \varphi x + \psi y - \frac{x-y}{2}(\varphi' x - \psi' y).$

34. $\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0 \therefore z = \phi(x + y\sqrt{-1}) + \psi(x + y\sqrt{-1}).$

35. $\frac{d^2z}{dx^2} - c^2\frac{d^2z}{dy^2} = 0 \therefore z = \phi(y - cx) + \psi(y + cx).$

36. $\frac{d^2z}{dx^2} - c^2\frac{d^2z}{dy^2} = xy \therefore$ . . . . . .

$z = \frac{x^3y}{6} + \phi(y - cx) + \psi(y + cx).$

37. $\frac{d^2z}{dx^2} = \frac{y^2}{x^2}\frac{d^2z}{dy^2} \therefore z = \varphi v + v^{\frac{1}{2}}\psi w$, where

$v = xy,\ w = \frac{y}{x}.$

38. $\frac{d^2z}{dx^2} - \frac{d^2z}{dxdy} - 2\frac{d^2z}{dy^2} = \frac{a}{y}.$

39. $\frac{d^2z}{dx^2} + \frac{3d^2z}{dxdy} + \frac{2d^2z}{dy^2} = x + y \therefore$ . . . . . .

$z = \frac{3x^2y - 2x^3}{6} + \varphi(y - x) + \psi(y - 2x).$

40. $\frac{d^2z}{dx^2} - a^2\frac{d^2z}{dy^2} + \frac{a}{y}\frac{dz}{dx} + \frac{a^2}{y}\frac{dz}{dy} = 0 \therefore$ . . . . .

$z = 2y\psi(y + ax) - \psi(y + ax) + \chi(y - ax).$

41. $x^2\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dxdy} + y^2\frac{d^2z}{dy^2} = 0 \therefore z = x\varphi\frac{y}{x} + \psi\frac{y}{x}.$

42. $\dfrac{d^2z}{dx^2} - m^2\dfrac{d^2z}{dy^2} = m\left\{\dfrac{dz}{dx} - m\dfrac{dz}{dy}\right\}$ ∴ . . . . . .

$z = \int e^{mx}\,dx\phi'(s - 2mx) + \psi s$, where $s = y + mx$.

43. $\dfrac{d^2z}{dx^2} = a^2\left\{\dfrac{d^2z}{dy^2} + \dfrac{2}{y}\dfrac{dz}{dy}\right\}$ ∴ . . . . . .

$yz = \phi(y + ax) + \psi(y - ax)$.

44. $\dfrac{d^2z}{dx^2} = a^2\left\{\dfrac{d^2z}{dy^2} - \dfrac{2z}{y}\right\}$ ∴ . . . . . .

$yz = \phi(y+ax) + \psi(y-ax) - \phi'(y+ax) - \psi'(y-ax)$.

45. $\dfrac{d^2z}{dx^2} + (m + n)\dfrac{d^2z}{dxdy} + mn\dfrac{d^2z}{dy^2} = \dfrac{x}{x+y}$ ∴ . . . .

$(m - n)z = \phi(y - mx) + \psi(y - nx)$ . . . .

$$+\frac{(m-n)xy}{(m+1)(n+1)} + \left\{\frac{ay^2}{2} + bxy + \frac{(2b-a)x^2}{2}\right\} l(x+y)$$

$$-\frac{ay^2}{4} - \frac{(2b-a)xy}{2}; \text{ where } a = \frac{1}{(m+1)^2} - \frac{1}{(n+1)^2}$$

and $b = \dfrac{n}{(n+1)^2} - \dfrac{m}{(m+1)^2}$.

CHAPTER IX.

The Integration of Fluxional Equations by Series and by Approximation.

THE METHOD OF APPROXIMATION.

1. When all other methods of integrating equations fail, we must endeavour to develope the dependent variable y in a series in terms of x.

This method has been already applied to two examples (vol. i. ch. 4, art. 11). If the law of the exponents be known, a series is to be assumed with indeterminate coefficients; this is to be differentiated, and being substituted in the proposed equation, the coefficients may be found by making each term $= 0$.

If the law of the exponents be unknown, there is frequently considerable difficulty in ascertaining it. From the same function, there may be sometimes deduced very different series, as has been already shown. We can in general succeed in the approximation only when x is either very small or very great: in the former case, we must endeavour to obtain an ascending, and in the latter, a descending series.

To obtain an ascending series, the indeterminate exponents must be assumed in an *increasing* arithmetick progression; and if the terms can be so arranged that both the coefficients and the exponents may be determined, there will result a series which converges the faster the less x is. If the exponents cannot be arranged in this manner, we may conclude that y cannot be developed in a series whose exponents increase in arithmetick progression; and recourse must be had to what is called the method of *Successive Substitutions*.

Both these methods are due to Sir I. Newton. Before we apply them to the integration of fluxional equations, we shall give a few instances of their application to algebraical functions, which may be sometimes more readily developed

by their means than by the theorems of Maclaurin and Lagrange.

For examples in which the law of the exponents is supposed known, vid. Alg. 348.

Ex. $y^3 + a^2y - 2a^3 + axy - x^3 = 0$.

To obtain an ascending series, assume $y = \text{A}x^\alpha + \text{B}x^{\alpha+\beta} + \text{C}x^{\alpha+2\beta} + \cdots$; then arranging the terms, we have

$$\left.\begin{array}{l} \text{A}^3x^{3\alpha} + 3\text{A}^2\text{B}x^{3\alpha+\beta} + 3\text{AB}^2x^{3\alpha+2\beta} + \text{B}^3x^{3\alpha+3\beta}\cdots \\ a^2\text{A}x^\alpha + a^2\text{B}x^{\alpha+\beta} + a^2\text{C}x^{\alpha+2\beta} + a^2\text{D}x^{\alpha+3\beta}\cdots \\ -2a^3 + a\text{A}x^{\alpha+1} + a\text{B}x^{\alpha+\beta+1} + a\text{C}x^{\alpha+2\beta+1}\cdots \\ \qquad\qquad - x^3 \end{array}\right\} = 0;$$ an

equation which is satisfied by assuming $\alpha = 0$, $\beta = 1$; in which case (Alg. 347. cor.) $\text{A}^3 + a^2\text{A} - 2a^3 = 0$; $3\text{A}^2\text{B} + a^2\text{B} + a\text{A} = 0$; $3\text{AB}^2 + a^2\text{C} + a\text{B} = 0$; $\cdots$ or

$$\text{A} = a,\ \text{B} = -\frac{1}{4},\ \text{C} = \frac{1}{64a},\ \cdots \text{ or}$$

$$y = a - \frac{x}{4} + \frac{x^2}{64a} - \frac{131x^3}{512a^2} + \cdots$$

If the exponents do not increase or decrease in an arithmetick progression and the law be altogether unknown, we must assume $y = \text{A}x^\alpha + \text{B}x^\beta + \text{C}x^\gamma + \cdots$ The following is a useful rule for finding the first term, whichever method of approximation may be adopted.

2. Sir I. Newton's rule for finding the first term of an ascending series.

Rule. 'Suppose the term of the proposed function where x is separately of the fewest dimensions to be $\text{D}x^m$; compare it successively with the other terms as with $\text{E}x^ty^s$; and observe where $\frac{m-t}{s}$ is found the greatest; then shall $\frac{m-t}{s}$ be the required exponent.'

For, let $\text{F}x^py^q$ be any other term of the function; then, substituting $\alpha = \frac{m-t}{s}$, we have (ex hyp.) $\alpha > \frac{m-p}{q}$, or $q\alpha + p > m$.

If therefore we assume an ascending series for y, viz.

$y = Ax^{\alpha} + Bx^{\beta} + \cdots$ where $\alpha = \frac{m-t}{s}$, in the limit we shall have $y = Ax^{\alpha}$; and if this be substituted in the proposed term $Fx^{v}y^{q}$, it becomes of the form $F'x^{q\alpha+p}$, which may be neglected when compared with Dx^{m}. The same is demonstrable of every other term of the function except $Ex^{t}y^{s}$, or its equal $E'x^{s\alpha+t}$, which does not vanish, since (ex hyp.) it is of the same dimensions as Dx^{m}. It appears then that $y = Ax^{\alpha}$ causes all the terms to vanish except two, which it reduces to the same dimensions; and by means of them A may be determined.

If α be assumed either $>$ or $< \frac{m-t}{s}$, there will be only one term left in the limit, and A cannot be determined. We conclude then, if y is developable in an ascending series of the form $Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} \cdots$ where B, C, $\cdots$ are functions of A which can be determined, that $\alpha = \frac{m-t}{s}$.

In some examples three or more terms of the functions remain in the limit, but the same rule obtains for determining both α and A.

Cor. 1. If it be observed where $\frac{m-t}{s}$ is found the *least*; it may be shown as in the article that $\frac{m-t}{s}$ is the exponent of the first term of a *descending* series.

Cor. 2. Similarly if any term of the function, as $Cx^{n}y^{r}$, be compared successively with the other terms, as with $Ex^{t}y^{s}$, and it be observed where $\frac{n-t}{s-r}$ is found the *greatest* or the *least*, it may be shown that the first term of the required series is Ax^{α}, where $\alpha = \frac{n-t}{s-r}$, and A is to be determined as before.

If both the numerator and the denominator of $\frac{n-t}{s-r}$ are negative, the reverse of this is true; for in multiplying both

sides of the symbols > and < by the same *negative* magnitude, the symbols are to be reversed.

Before this rule is applied, the example should be cleared of fractions, and if any of the exponents are fractions, they must be made integers by substitution.

Sir I. Newton has also invented for the same purpose a rectangular diagram, which the reader will find explained in The Method of Fluxions, p. 9.

3. *Given an implicit function; required to develope the dependent variable in an ascending series of the independent by the method of successive substitutions.*

Let $y = Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} \cdots$ be the required series. Having determined Ax^{α} by the above rule, an approximate value of y may be obtained when x is small.

To obtain a nearer value; let $y = Ax^{\alpha} + y'$, where $y' = Bx^{\beta} + Cx^{\gamma} + \cdots =$, in the limit, Bx^{β}. Substituting this value of y' there results a function from which the first term of y' or Bx^{β} may be determined as before; and we thus obtain the value of $y = Ax^{\alpha} + Bx^{\beta}$.

Similarly, substituting $y' = Bx^{\beta} + y''$, Cx^{γ} will result; and by repeating the process, the series may be calculated to as many times as may be required.

Ex. 1. To develope $y^3 + a^2y - 2a^3 + axy - x^3 = 0$ in an ascending series.

Since 0 is the least exponent of x, the quotients are $0, 0, -1, -\frac{3}{0}$; of which 0 is the greatest; and the equation for determining Ax^{α}, or that which arises from substituting $y = Ax^0$, and neglecting the terms in which x enters, is $y^3 + a^2y - 2a^3 = 0$, which shows that $y = a$ when x is indefinitely diminished.

Let $y = a + y'$; then by substitution $4a^2y' + 3ay'^2 + y'^3 + a^2x + axy' - x^3 = 0$ (1); where the quotients are $1, \frac{1}{2}, \frac{1}{3}, 0, -\frac{2}{0}$ or $\beta = 1$; and to find B, we have, neglecting the terms in which x^2 enters

$4a^2y' + a^2x = 0$ or $y' = -\frac{x}{4}$; and a nearer value of y is

$$y = a - \frac{x}{4}.$$

Next, let $y' = -\frac{x}{4} + y''$; and if this be substituted in

(1), it will appear that $Bx^{\beta} = \frac{x^2}{64a}$; and so on.

Ex. 2. $ax^3 + x^3y - ay^3 = 0$.

The quotients are 0 and 1; and the equation for determining Ax^{α} is $ax^3 - ay^3 = 0$, or $y = x$.

Let $y = x + y'$; then, by substitution,
$x^4 + x^3y' - 3ax^2y' - 3axy'^2 - ay'^3 = 0$ (1), where the quotients are 1, 2, $\frac{3}{2}$, $\frac{4}{3}$; and the equation for determining Bx^{β} is $x^4 - 3ax^2y' = 0 \therefore y'$ or $Bx^{\beta} = \frac{x^2}{3a}$.

Next, let $y' = \frac{x^2}{3a} + y''$; then (1) becomes

$$\left.\begin{aligned}\frac{x^6}{27a^2} + 3ax^2y'' + x^3y'' + \frac{x^4y''}{3a} \\ + 3axy''^2 + x^2y''^2 + ay''^3\end{aligned}\right\} = 0,$$ where the quotients are 4, 3, 2, $\frac{5}{2}$, or $\gamma = 4$; and to find c, we have

$\frac{x^6}{27a^2} + 3ax^2y'' = 0$ or $y'' = -\frac{x^4}{81a^3}$; $\therefore$

$y = x + \frac{x^2}{3a} - \frac{x^4}{3^4a^3} + \frac{x^5}{3^5a^4} - \cdots$

To obtain a *descending* series; $\alpha = 0$, and to find A, we have $ax^3 + x^3y = 0$ or $y = -a$.

Let $y = -a + y'$; then, by substitution,
$x^3y' + a^4 - 3a^3y' + 3a^2y'^2 - ay'^3 = 0$; where the quotients are -3, 0; $\therefore \beta = -3$; and to find B, we have
$x^3y' + a^4 = 0$ or $y' = -a^4x^{-3}$; and, repeating the process, there will result $y = -a - a^4x^{-3} - 3a^7x^{-6} - 12a^{10}x^{-9} - \cdots$

To find other series, compare the second term with the first and third; then (cor. 2.) the quotients are -0 and 1, the same as before. But if the third term be compared with the first and second, the quotients are 1 and $\frac{3}{2}$; and substituting $y = Ax^{\frac{3}{2}}$, the second and third terms are of the same dimensions and greater than the first; or $Ax^{\frac{3}{2}}$ is the first term of a *descending* series; and $Ax^{\alpha} = \pm \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$.

Let $y = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}} + y'$; then, by substitution,

$ax^3 - 2x^3y' - 3a^{\frac{1}{2}}x^{\frac{3}{2}}y'^2 - ay'^3$; where the quotients are

$0, \frac{3}{4}, 1 \therefore \beta = 0$, and to determine Bx^{β}, we have

$ax^3 - 2x^3y' = 0$, or $y' = \frac{a}{2}$.

Similarly, assume $y' = \frac{a}{2} + y''$, and there will result

$$y = \pm \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{a}{2} \mp \frac{3a^{\frac{5}{2}}}{8} x^{-\frac{3}{2}} + \frac{a^4}{2} x^{-3} \mp \cdots$$

This developement may be also obtained by the method of Indeterminate Coefficients, by substituting

$y = Ax^{\alpha} + Bx^{\alpha-\beta} + Cx^{\alpha-2\beta} + Dx^{\alpha-3\beta} + \cdots$

Ex. 3. $y^3 - xy^{\frac{1}{2}} + x^{\frac{4}{3}} = 0$.

Substitute $v = x^{\frac{1}{3}}$, $w = y^{\frac{1}{2}}$ $\Big\}$ $\therefore w^6 - v^3w + v^4 = 0$.

To develope this by the method of Indeterminate Coefficients; the first term of an ascending series is v. Assume then $w = v + Bv^{1+\beta} + Cv^{1+2\beta} + Dv^{1+3\beta} \cdots \therefore$ we have

$$\left.\begin{array}{l} v^6 + 6Bv^{6+\beta} + 6Cv^{6+2\beta} \cdots \\ - v^4 - Bv^{4+\beta} - Cv^{4+2\beta} - Dv^{4+3\beta} \cdots \\ + v^4 \end{array}\right\} = 0 \therefore \beta = 2,$$

$B = 1$, $C = 6$, $D = 6.6$, $\cdots$; or $w = v + v^3 + 6v^5 + 6.6v^7 \cdots$ and replacing x and y, $y^{\frac{1}{2}} = x^{\frac{1}{3}} + x + 6x^{\frac{5}{3}} + 6.6x^{\frac{7}{3}} \cdots$

When the equation consists of many terms, this method of finding all the possible developements becomes laborious; we shall therefore give another method of finding all the possible first terms; and the others are to be obtained as in the article.

(1) (2) (3) (4) (5) (6) (7)

Ex. 4. $x^2y^2 + axy^2 + bx^2y + cx^3 + exy + fx^2 + gy = 0$.

When $x = \frac{1}{\infty}$;

(1) may be neglected compared with (2)
(3) . . . . . (5)
(4) . . . . . (6)
(5) . . . . . (7) $\Big\}$ $\therefore$

$axy^2 + fx^2 + gy = 0$.

In which equation y must be either finite, ∞, or $\frac{1}{\infty}$.

Now y cannot be finite; for then we should have $gy = 0$.

Suppose $y = \infty$; then $axy^2 + gy = 0$ or $axy + g = 0$.

Suppose $y = \frac{1}{\infty}$; then axy^2 may be neglected compared with gy or $fx^2 + gy = 0$.

When $x = \infty$;

(2) may be neglected compared with (1)
(5) . . . . . (3)
(6) . . . . . (4)
(7) . . . . . (3)

$\therefore$

$x^2y^2 + bx^2y + cx^3 = 0$, or $y^2 + by + cx = 0$.

Here y must $= \infty$; and the only possible equation in the limit is $y^2 + cx = 0$.

It appears then that this example admits of only *three* developements; the two ascending series are

$$y = -\frac{f}{g}x^2 + \frac{ef-cg}{g^2}x^3 - \frac{e^2f-ceg-bgf}{g^3}x^4 + \cdots$$
$$y = -\frac{g}{a}x^{-1} - \frac{ae-g}{a^2} - \frac{a^2b-ae+g}{a^3}x - \cdots$$

The descending series is of the form $y = \pm c^{\frac{1}{2}}x^{\frac{1}{2}} + \text{B}$; and consequently the curve to which the equation belongs has an asymptote which is a parabola.

Ex. 5. $x^7 - a^3x^3y + a^3x^2y^2 + a^5y^2 - 2a^5xy + a^5x^2 = 0$.

Or $x^7 - x^3y + x^2y^2 + y^2 - 2xy + x^2$; for a may be at any time restored by rendering the terms homogeneous.

When $x = \frac{1}{\infty}$; (1), (2), (3) may be neglected, and there results $y^2 - 2xy + x^2 = 0$; or there are two ascending series whose first terms each $= x$.

When $x = \infty$; the equation becomes $x^7 - x^3y + x^2y^2 = 0$, or $x^5 - xy + y^2 = 0$; where y is necessarily infinite and of a higher order than x, or $x^5 + y^2 = 0$, which is possible only when x is negative.

To find the ascending series; substitute $y = x + y'$ $\therefore$ $x^7 + x^3y' + y'^2 = 0$ (α), where the quotients are 4, $\frac{7}{2}$; $\therefore$ $x^7 + x^3y' = 0$, or $y' = -x^4$.

Next let $y' = -x^4 + y''$; then, by substitution, $x^3y'' + x^8 - 2x^4y'' + y''^2 = 0$, where the quotients are 5, 4; $\therefore$ $x^3y'' + x^8 = 0$, or $y'' = -x^5$; and so on.

To find the other ascending series; compare in (α) x^3y' with the other terms, and the quotients are -4, 3; and $x^3y' + y'^2 = 0$, or $y' = -x^3$.

Let $y' = -x^3 + y''$; then, by substitution, $x^7 - x^3y'' + y''^2 = 0$; where the quotients are 4, $\frac{7}{2}$; $\therefore$ $x^7 - x^3y'' = 0$, or $y'' = x^4$.

Next, let $y'' = x^4 + y'''$; then we have $-x^3y''' + x^8 + 2x^4y''' + y'''^2 = 0$; where the quotients are 5, 4, $\therefore$ $-x^3y''' + x^8 = 0$, or $y''' = x^5$; wherefore the required series are

$$y = x - \frac{x^4}{a^3} - \frac{x^5}{a^4} \cdots$$

$$y = x - \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{x^5}{a^4} + \cdots$$

Ex. 6. Required to find the asymptotes of the curve $y = \frac{ax^4+bx^3+cx^2+ex+g}{mx^3+nx^2+px+q}$.

When $x = \infty$; $y = \frac{ax^4}{mx^3} = \frac{ax}{m}$; or the equation admits of only one descending series.

Let $y = \text{A}x^\alpha + \text{B}x^{\alpha-\beta} + \text{C}x^{\alpha-2\beta} + \cdots$; then

$$\left.\begin{array}{llll} m\text{A}x^{\alpha+3} & + m\text{B}x^{\alpha-\beta+3} & + m\text{C}x^{\alpha-2\beta+3} & + \cdots \\ & + n\text{A}x^{\alpha+2} & + n\text{B}x^{\alpha-\beta+2} & + \cdots \\ & & + p\text{A}x^{\alpha+1} & + \cdots \\ - ax^4 & - bx^3 & - cx^2 & - \cdots \end{array}\right\} = 0;$$

$\therefore$ $\alpha = 1$ and $\beta = 1$.

Also, $m\text{A} - a = 0$; $m\text{B} + n\text{A} - b = 0$; $\therefore$ $\text{A} = \frac{a}{m}$; $\text{B} = \frac{mb-na}{m^2}$; or the curve has only one asymptote; viz. the right line $y = \frac{ax}{m} + \frac{mb-na}{m^2}$.

The curve has as many *infinite ordinates* as there are possible roots in the equation $mx^3 + nx^2 + px + q = 0$.

4. *Continued fractions.* Similar to the preceding is the method of continued fractions. For, $\text{A}x^\alpha$ being the first term, assume successively $y = \frac{\text{A}x^\alpha}{1+y'}$, $y' = \frac{\text{B}x^\beta}{1+y''}$, $y'' = \frac{\text{C}x^\gamma}{1+y'''} \cdots$; then if α, β, $\gamma \cdots$ be an ascending series, approximate values of y, when x is indefinitely diminished, are $y = \text{A}x^\alpha$,

$y = \frac{Ax^\alpha}{1+Bx^\beta}$, $y = \frac{Ax^\alpha}{1+\frac{Bx^\beta}{1+Cx^\gamma}}$, &c. = &c.; each of which is more accurate than that which precedes it.

The value of Bx^β is determined by substituting $y = \frac{Ax^\alpha}{1+y'}$ in the proposed function, and taking the value of y in the limit; and in the same manner the succeeding terms may be found.

If an approximate value of y be required when x is indefinitely increased, Ax^α must be the first term of a descending series.

Ex. Required to develope e^x in a continued fraction.

Let $y = e^x = 1 + x + \frac{x^2}{1.2} + \cdots$; then $y = 1 + x$ is a first approximation.

Assume $y = 1 + \frac{x}{1+y'}$ which, substituted in the proposed function, gives $\frac{x}{1+y'} = x + \frac{x^2}{1.2} + \cdots$ or

$1 + y' = \left(1 + \frac{x}{2} + \cdots\right)^{-1} =$, in the limit, $1 - \frac{x}{2} \therefore \cdots$

$y = 1 + \frac{x}{1 - \frac{x}{2}}$ is a second approximation.

Assume y' or $-\frac{x}{2} = \frac{-\frac{x}{2}}{1+y''} \therefore 1 + \frac{-\frac{x}{2}}{1+y''} =, 1 + y', \cdots$

$= \left(1 + \frac{x}{2} + \cdots\right)^{-1} = 1 - \frac{x}{2} + \frac{x^2}{3.4} - \cdots$ or

$1 + y'' = \left(1 - \frac{x}{3.2} + \cdots\right)^{-1} =$, in the limit, $1 + \frac{x}{3.2} \therefore$

$y = 1 + \frac{x}{1 - \frac{x}{2}}$ is a third approximation.

$\overline{1 + \frac{x}{3.2}}$.

By repeating the process, it may be shown that

$$y = 1 + \cfrac{x}{1 - \cfrac{\frac{x}{1.2}}{1 + \cfrac{\frac{x}{3.2}}{1 - \cfrac{\frac{x}{3.2}}{1 + \cfrac{\frac{x}{5.2}}{1 - \cdots}}}}}$$

5. PRAXIS.

1. $y^3 - by^2 + 9bx^2 - x^3 = 0 \therefore y = x - \frac{8b}{3} - \frac{16b^2}{3^2 x} - \cdots$

2. $\frac{y^5}{5} - \frac{y^4}{4} + \frac{y^3}{3} - \frac{y^2}{2} + \frac{y}{1} = x - a = z \therefore \cdots$

$$y = z + \frac{z^2}{2} + \frac{z^3}{3 \times 2} + \frac{z^4}{3.4 \times 2} + \frac{z^5}{3.4.5 \times 2} + \cdots$$

3. $x^2y^3 + 3ax^2y^2 + 3a^2x^2y - a^3xy - a^4x + a^5 = 0$ may be developed in three descending series, first term of each $= -a$.

4. $6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5x^2 - 3a^5xy + a^5y^2 = 0 \therefore$

$$\left.\begin{aligned} y &= x + \frac{3x^2}{a^2} + \frac{15x^5}{a^4} + \frac{4x^6}{a^5} \cdots \\ y &= 2x - \frac{4x^3}{a^2} - \frac{16x^5}{a^4} \cdots \end{aligned}\right\}.$$

5. $a - bx^3y + \frac{cy^2}{x} - \frac{ey^4}{x^5} + fx^2y^5 - \frac{gy^6}{x^3} = 0 \therefore \cdots$

$$\left.\begin{aligned} y &= \frac{a^{\frac{1}{4}}}{e^{\frac{1}{4}}} x^{\frac{5}{4}} + \text{B}x^{\frac{5}{4}} + \cdots \\ y &= \sqrt{-\frac{e}{f}}\, x^{-3} + \text{B}x^{-3} \cdots \end{aligned}\right\}.$$

$$6.\ (1+x)^n = 1 + \cfrac{nx}{1 - \cfrac{(n-1)x}{1.2}}$$

$$\cfrac{}{1 + \cfrac{(n+1)x}{3.2}}$$

$$\cfrac{}{1 - \cfrac{(n-2)x}{3.2}}$$

$$\cfrac{}{1 + \cfrac{(n+2)x}{5.2}}$$

$$\cfrac{}{1 - \cdots}$$

6. *Required to integrate in a series* $dy + ydx = mx^n dx$.

Assume $y = Ax^\alpha + Bx^{\alpha+1} + Cx^{\alpha+2} + \cdots$; then

$$\frac{dy}{dx} = \alpha A x^{\alpha-1} + (\alpha+1)\beta x^\alpha + (\alpha+2)Cx^{\alpha+1}\cdots$$

Substituting these, and arranging the terms, we have

$$\left.\begin{array}{llll} \alpha A x^{\alpha-1} + & (\alpha+1)Bx^\alpha + & (\alpha+2)Cx^{\alpha+1}\cdots \\ -mx^n + & Ax^\alpha + & Bx^{\alpha+1}\cdots \end{array}\right\} = 0;$$ which

may be satisfied by assuming $\alpha - 1 = n$, in which case $\alpha A - m = 0$; $(\alpha+1)B + A = 0$; $(\alpha+2)C + B = 0$; $\cdots$

or $A = \dfrac{m}{\alpha} = \dfrac{m}{n+1}$; $B = -\dfrac{A}{\alpha+1} = \dfrac{-m}{(n+1)(n+2)}$; $\cdots$

$C = \dfrac{m}{(n+1)(n+2)(n+3)}$; $\cdots$ whence

$$\frac{y}{m} = \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{(n+1)(n+2)} + \frac{x^{n+3}}{(n+1)(n+2)(n+3)} - \cdots$$

Since this method does not introduce an arbitrary constant, it cannot give the complete primitive, whenever, as in this example, the constant enters as a coefficient of some function of the variable. The following method has been devised by Euler to supply this defect. It does not introduce the arbitrary constant itself, but two corresponding values of x and y by means of which it may be determined. (Vol. i. ch. 2, 6.)

7. *Required to integrate an equation by a series so as to introduce the arbitrary constant.*

Let a and b be two corresponding values of x and y. Substitute $x = a + t$, $y = b + v$, where v and t vanish together. Assume a series for v in terms of t with indeterminate coefficients and exponents; and arranging the terms, make such assumptions that each line of terms may vanish when $t = 0$: then, if by means of these assumptions, the series can be determined, there will result v or $y - b$ in terms of t or $x - a$; which is the *complete* value of y.

The series may be also found by means of Euler's Theorem, vol. i. ch. 4, 45; in which a and b must be substituted for x and y in the values of the fluxional coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \cdots$

Ex. 1. $dy + ydx = mx^n dx$, or $\frac{dy}{dx} = mx^n - y$.

By substitution the proposed becomes

$$\frac{dv}{dt} = m(a + t)^n - (b + v).$$

Assume $v = At^\alpha + Bt^{\alpha+1} + Ct^{\alpha+2} + \cdots$; then

$$\frac{dv}{dt} \text{ or } m(a + t)^n - (b + v) = \alpha At^{\alpha-1} + \cdots$$

$(\alpha + 1)\beta t^\alpha + (\alpha + 2)Ct^{\alpha+1}$; and arranging the terms, we have

$$\left.\begin{array}{l} \alpha At^{\alpha-1} + (\alpha+1)Bt^\alpha + (\alpha+2))Ct^{\alpha+1} + \cdots \\ +b \quad + At^\alpha + Bt^{\alpha+1} + \cdots \\ -ma^n - mna^{n-1}t - \dfrac{mn(n-1)a^{n-2}t^2}{2} + \cdots \end{array}\right\} = 0.$$

In order to satisfy this, we must have $\alpha - 1 = 0$ or $\alpha = 1$; in which case, $\alpha A + b - ma^n = 0$;

$$(\alpha+1)B + A - mna^{n-1} = 0; \ (\alpha+2)C + B - \frac{mn(n-1)a^{n-2}}{2} = 0;$$

$$\text{or } A = ma^n - b; \ B = -\frac{ma^n - mna^{n-1} - b}{1.2};$$

$$C = \frac{ma^n - mna^{n-1} + mn(n-1)a^{n-2} - b}{1.2.3}; \cdots \therefore$$

$$y-b=(ma^n-b)(x-a)-\frac{ma^n-mna^{n-1}-b}{1.2}(x-a)^2+\cdots$$

Ex. 2. $ydx+ydy=dx$ or $y\left(1+\frac{dy}{dx}\right)=1$.

The law of the exponents being unknown, assume $y=Ax^\alpha+Bx^\beta+Cx^\gamma+\cdots$; then

$\frac{dy}{dx}=\alpha Ax^{\alpha-1}+\beta Bx^{\beta-1}+\gamma Cx^{\gamma-1}+\cdots$; and

$(Ax^\alpha+Bx^\beta+Cx^\gamma+\cdots)\times\ .\ .\ .\ .\ .\ .\ .\ .\ .$

$\{1+\alpha Ax^{\alpha-1}+\beta Bx^{\beta-1}+\gamma Cx^{\gamma-1}+\cdots\}=1$; and arranging the terms, we have

$$\left.\begin{array}{llllll} \alpha A^2x^{2\alpha-1}+(\alpha+\beta)ABx^{\alpha+\beta-1}+(\alpha+\gamma)ACx^{\alpha+\gamma-1}+\cdots \\ -1 \quad + \quad Ax^\alpha \quad + \quad \beta B^2x^{2\beta-1} \quad +\cdots \\ + \quad Bx^\beta \quad +\cdots \end{array}\right\}=0.$$

This is satisfied by assuming $2\alpha-1=0$; $\alpha+\beta-1=\alpha$;

$\alpha+\gamma-1=\beta;\cdots$ or $\alpha=\frac{1}{2}$, $\beta=1$, $\gamma=\frac{3}{2}$, $\delta=2;\cdots$

in which case we have $\frac{A^2}{2}-1=0$; $\frac{3AB}{2}+A=0$;

$2AC+B^2+B=0;\cdots$ or $A=\sqrt{2}$; $B=-\frac{2}{3}$; $\cdots\therefore$

$$y=\sqrt{2}x^{\frac{1}{2}}-\tfrac{2}{3}x+\frac{\sqrt{2}}{18}x^{\frac{3}{2}}-\cdots$$

If the complete primitive be required; substitute $x=a+t$, $y=b+v$; and the proposed becomes

$$(b+v)\left\{1+\frac{dv}{dt}\right\}=1.$$

Assume $v=At^\alpha+Bt^\beta+Ct^\gamma+\cdots$; then, by substitution, we have $(b+At^\alpha+Bt^\beta+Ct^\gamma+\cdots)\times\ .\ .\ .\ .\ .\ .$

$\{1+\alpha At^{\alpha-1}+\beta Bt^{\beta-1}+\gamma Ct^{\gamma-1}+\cdots\}=1$; and arranging the terms,

$$\left.\begin{array}{llllll} -1+\alpha A^2t^{2\alpha-1}+(\alpha+\beta)ABt^{\alpha+\beta-1}+(\alpha+\gamma)ACt^{\alpha+\gamma-1}\ldots \\ +b+At^\alpha \quad + \quad Bt^\beta \quad + \quad Ct^\gamma \ \ldots \\ +b\alpha At^{\alpha-1}+b\beta Bt^{\beta-1}+b\gamma Ct^{\gamma-1}+ \quad b\delta Dt^{\delta-1}\ldots \\ + \quad \beta B^2t^{2\beta-1}\ldots \end{array}\right\}=0.$$

Whence $\alpha=1$; $\beta=2$; $\gamma=3$; $\cdots$ Also $b\text{A}+b-1=0$; $2b\text{B}+\text{A}+\text{A}^2=0$; $3b\text{C}+\text{B}+3\text{AB}=0$; $4b\text{D}+\text{C}+4\text{AC}+2\text{B}^2=0$; $\cdots$ from which A, B, C, $\cdots$ may be found, and the value of v or $y-b$ obtained in terms of t or $x-a$.

Otherwise:

$$\frac{dy}{dx}=\frac{1}{y}-1=\frac{1-y}{y}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{y^2}\frac{dy}{dx}=\frac{1}{y^2}-\frac{1}{y^3}=\frac{y-1}{y^3}$$

$$\frac{d^3y}{dx^3}=\left\{-\frac{2}{y^3}+\frac{3}{y^4}\right\}\frac{dy}{dx}=\frac{(y-1)(2y-3)}{y^5}$$

$$\cdots=\cdots$$

$$\therefore y-b=\frac{1-b}{b}(x-a)-\frac{1-b}{1.2b^3}(x-a)^2 \quad \cdots\cdots$$

$$+\frac{(1-b)(3-2b)}{1.2.3b^5}(x-a)^3-\cdots$$

8. The above method is also applicable to equations of a higher order than the first; in which case it sometimes has the advantage of giving the complete primitive at once, without substituting new functions for the variable.

Ex. $d^2y+ax^ny{dx}^2=0$.

Here x and dx are of 0 dimensions in the first term, and of $n+2$ in the second; from which it may be inferred that the exponents increase or decrease by $n+2$.

Assume $y=\text{A}x^\alpha+\text{B}x^{\alpha+n+2}+\text{C}x^{\alpha+2n+4}\cdots$ then

$$\frac{d^2y}{dx^2}=\alpha(\alpha-1)\text{A}x^{\alpha-2}+(\alpha+n+2)(\alpha+n+1)\text{B}x^{\alpha+n}$$

$+(\alpha+2n+4)(\alpha+2n+3)\text{C}x^{\alpha+2n+2}$, and the terms must be arranged in the following manner;

$$\left.\begin{matrix}\alpha(\alpha-1)\text{A}x^{\alpha-2}+(\alpha+n+2)(\alpha+n+1)\text{B}x^{\alpha+n}\\ +\qquad\qquad\qquad a\text{A}x^{\alpha+n}\end{matrix}\right\}=0.$$

Whence $\alpha(\alpha-1)\text{A}=0$; $(\alpha+n+2)(\alpha+n+1)\text{B}+a\text{A}=0\cdots$

This may be satisfied by supposing A an indeterminate quantity; and either $\alpha=0$, or $\alpha-1=0$.

Let A and A′ be the values of A corresponding to these

two suppositions; then, when $\alpha = 0$, we have

$$B = -\frac{aA}{(n+1)(n+2)},\ C = \frac{a^2A}{(n+1)(n+2)(2n+3)(2n+4)} \cdots$$

Also, when $\alpha - 1 = 0$ or $\alpha = 1$, we have

$$B' = -\frac{aA'}{(n+2)(n+3)},\ C' = \frac{a^2A'}{(n+2)(n+3)(2n+4)(2n+5)} \cdots$$

from which we obtain two conjugate values of y, each of which satisfies the proposed; and consequently (ch. 6. art. 20.) the complete primitive is their sum; or

$$y = A\left\{1 - \frac{ax^{n+2}}{(n+1)(n+2)} + \frac{a^2x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)} \cdots - \frac{a^3x^{3n+6}}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} + \cdots\right\}$$
$$+ A'\left\{x - \frac{ax^{n+3}}{(n+2)(n+3)} + \frac{a^2x^{2n+5}}{(n+2)(n+3)(2n+4)(2n+5)} \cdots - \frac{a^3x^{3n+7}}{(n+2)(n+3)(2n+4)(2n+5)(3n+6)(3n+7)} + \cdots\right\}.$$

There is no descending series.

Cor. If the proposed equation be of the third order, we shall have $\alpha(\alpha - 1)(\alpha - 2)A = 0$; and making three suppositions with respect to α, a complete value of y may be had containing three arbitrary constants.

9. The first of the above series fails when either $n+2=0$ or $in + 2i - 1 = 0$, *i. e.* when $n = -2$ or $= -\frac{2i-1}{i}$; and the second, when $n = -2$ or $= -\frac{2i+1}{i}$; and consequently they both fail only when $n = -2$.

In this case the equation is $d^2y + ax^{-2}ydx^2 = 0$, which is homogeneous, and may be integrated as in ch. 6. 30, ex. 1; or in the following manner.

Substitute $y = x^\alpha$; then we have $\alpha(\alpha - 1) + a = 0$. Let the roots of this be α and α', then $y' = x^\alpha$ and $y'' = x^{\alpha'}$; wherefore the complete primitive is $y = Ax^\alpha + A'x^{\alpha'}$, where $\alpha = \frac{1 + \sqrt{1-4a}}{2}$ and $\alpha' = \frac{1 - \sqrt{1-4a}}{2}$.

If $a > \frac{1}{4}$; let $\alpha = \frac{1}{2} + \beta\sqrt{-1}$, $\alpha' = \frac{1}{2} - \beta\sqrt{-1}$; then

$y = x^{\frac{1}{2}}(\text{A}x^{\beta\sqrt{-1}} + \text{A}'x^{-\beta\sqrt{-1}})$

But $x^{\beta\sqrt{-1}} = e^{lx.\beta\sqrt{-1}} = \cos.\beta lx + \sqrt{-1}\sin.\beta lx$

$x^{-\beta\sqrt{-1}} = e^{-lx.\beta\sqrt{-1}} = \cos.\beta lx - \sqrt{-1}\sin.\beta lx$

(Vol. i. ch. 2. 36.); whence

$y = x^{\frac{1}{2}}\{(\text{A} + \text{A}')\cos.\beta lx + (\text{A} - \text{A}')\sqrt{-1}\sin.\beta lx \cdots$

$= cx^{\frac{1}{2}}(\sin.\beta lx + c')\}$ where $\beta = \frac{\sqrt{4a-1}}{2}$.

If $a = \frac{1}{4}$, the roots of the quadratick are equal and $\beta = 0$; whence $y = x^{\frac{1}{2}}(\text{A} + \text{A}') = cx^{\frac{1}{2}}$, which is only a particular primitive.

To obtain the complete primitive, proceeding as in ch. 6. 28. cor. 2, substitute $y = vy'$, where $y' = x^{\frac{1}{2}}$; then the proposed becomes $v(d^2y' + ax^{-2}y'dx^2) + y'd^2v + 2dvdy' = 0$; wherefore $y'd^2v + 2dvdy' = 0$, or $\frac{d^2v}{dv} + \frac{2dy'}{y'} = 0$; and integrating, $y'^2dv = c'dx$ or $dv = \frac{c'dx}{x}$; and again integrating, $v = c'lx$; or the required primitive is

$y = cx^{\frac{1}{2}} + c'x^{\frac{1}{2}}lx = x^{\frac{1}{2}}(c + c'lx)$.

10. When $n = -\frac{2i \mp 1}{i} = -2i \pm \frac{1}{i}$, one of the series of art. 8 fails, viz. the first when the sign is + and the second when it is −. In either case then we obtain only a particular value of y; and if from this we deduce the complete value by substituting as before $y = vy'$, the result will be $v = c'\int\frac{dx}{y'}$, and consequently $y = y'\left\{c + c'\int\frac{dx}{y'}\right\}$.

This value of y is under a very inconvenient form, inasmuch as $\int\frac{dx}{y'}$ contains a series in the denominator. Euler, to obviate this difficulty, makes use of a very peculiar transformation. He observed that one of the series which express the value of y fails whenever the primitive contains lx; and he attributes the failure to the property of the function lx, that its fluxional coefficients become infinite when $x = 0$. He therefore substitutes $y = p + qlx$, p and q

being two indeterminate quantities. By certain assumptions, p and q and consequently the value of y, may be determined, as will be seen in the next article.

In order to introduce a constant, we may substitute $y = p + cq + qlx = p + ql.cx$.

11. *Required to integrate* $d^2y + ax^n y dx^2 = 0$ *when* $n = -\frac{2i \pm 1}{i}$.

Substitute $y = p + cq + qlx = p + ql.cx$; then the proposed becomes $d^2p + \frac{2dxdq}{x} - \frac{qdx^2}{x^2} + apx^n dx^2 \cdot \cdot \cdot \cdot$

$+ (d^2q + ax^n q dx^2) l.cx = 0$.

The terms of this equation which involve lx may be assumed to destroy each other; and consequently we have

$d^2q + ax^n q dx^2 = 0$ (1); and $\frac{d^2p}{dx^2} + \frac{2}{x}\frac{dq}{dx} - \frac{q}{x^2} + apx^n = 0$ (2).

Of these (1) is of the same form as the proposed; and consequently we can find a particular value of q. Substituting this in (2), and integrating it in a series, we shall find p and therefore y in a series in terms of x.

Ex. 1. $d^2y + \frac{aydx^2}{x} = 0$.

Here $n = -1$, and the series which does not fail is the second.

Assume then $q = \text{A}'\left\{x - \frac{ax^2}{1^2.2} + \frac{a^2x^3}{1^2.2^2.3} - \frac{a^3x^4}{1^2.2^2.3^2.4} + \cdots\right\}$;

which for the sake of convenience represent by

$q = \text{A}'x + \text{B}'x^2 + \text{C}'x^3 + \cdots$

Assume $p = \text{A} + \text{B}x + \text{C}x^2 + \cdots$; then substituting these values of p and q in (2), arrange the terms in the following manner.

$$\left.\begin{array}{r} 2\text{C} + 3.2\text{D}x + \cdots \\ 2(\text{A}'x^{-1} + 2\text{B}' + 3\text{C}'x + \cdots) \\ -\text{A}'x^{-1} - \text{B}' - \text{C}'x - \cdots \\ a\text{A}x^{-1} + a\text{B} + a\text{C}x + \cdots \end{array}\right\} = 0;$$ which gives, if B be assumed $= 0$, $\text{A}' + a\text{A} = 0$; $2\text{C} + 3\text{B}' + a\text{B} = 0$; $2.3\text{D} + 5\text{C}' + a\text{C} = 0$; $\cdots$ or

$\text{A} = -\frac{\text{A}'}{a}$; and B is not determined;

$$\text{C} = -\frac{3\text{B}'}{2} - \frac{a\text{B}}{1.2} = +\frac{3a\text{A}'}{1^2.2^2} - \frac{a\text{B}}{1.2}$$

$$\text{D} = -\frac{5\text{C}'}{6} - \frac{a\text{C}}{2.3} = -\frac{5a^2\text{A}'}{1^2.2^2.3^2} - \frac{a\text{C}}{2.3}$$

$$\text{E} = -\frac{7\text{D}'}{12} - \frac{a\text{D}}{3.4} = +\frac{7a^3\text{A}'}{1^2.2^2.3^2.4^2} - \frac{a\text{D}}{3.4}$$

$$\text{F} = -\frac{9\text{E}'}{20} - \frac{a\text{E}}{4.5} = -\frac{9a^4\text{A}'}{1^2.2^2.3^2.4^2.5^2} - \frac{a\text{E}}{4.5}$$

$\ldots = \ldots$

In this series we may suppose $\text{B} = 0$; because the term $\text{B}x$, which we shall neglect by this supposition, may be incorporated into that part of the value of y which is represented by cq.

Having thus found the values of p and q, y will result from substituting them in $y = p + ql.cx$, which contains two arbitrary constants A' and c.

$$\text{Thus } y = \text{A}'\left\{-\frac{1}{a} + \frac{3a}{1^2.2^2} - \frac{14a^2}{1^3.2^3.3^2} - \cdots\right\}$$

$$+ \text{A}'l.cx\left\{x - \frac{ax^2}{1.2} + \frac{a^2x^3}{1^2.2^2.3} - \cdots\right\}.$$

If B remains undetermined, we must substitute $y = p + qlx$; and the resulting value of y will be the same as before, making $\text{B} = \text{A}'lc$.

Ex. 2. $d^2y + ayx^{-\frac{3}{2}}dx^2 = 0$.

The series which does not fail is the second.

$$\text{Assume } q = \text{A}'\left\{x - \frac{2^2a}{1.3}x^{\frac{3}{2}} + \frac{2^4a^2}{1.2^2.3}x^2 - \frac{2^6a^3}{1.2^2.3^2.4}x^{\frac{5}{2}} + \ldots\right\}$$

$$\text{or} = \text{A}'x + \text{B}'x^{\frac{3}{2}} + \text{C}'x^2 + \cdots$$

In order to integrate in a series (2), which is

$\frac{d^2p}{dx^2} + \frac{2}{x}\frac{dq}{dx} - \frac{q}{x^2} + \frac{ap}{x^{\frac{3}{2}}} = 0$, assume

$p = \text{A} + \text{B}x^{\frac{1}{2}} + \text{C}x + \text{D}x^{\frac{3}{2}} + \cdots$; then

$$\left.\begin{aligned} &-\frac{\text{B}}{4}x^{-\frac{3}{2}}+0.x^{-1}+\frac{3\text{D}}{4}x^{-\frac{1}{2}}+2\text{E}+\frac{15\text{F}}{4}x^{\frac{1}{2}}+\cdots \\ &\quad+2(\text{A}'x^{-1}+\frac{3\text{B}'}{2}x^{-\frac{1}{2}}+2\text{C}'+\frac{5\text{D}'}{2}x^{\frac{1}{2}}+\cdots) \\ &\quad-\text{A}'x^{-1}-\text{B}'x^{-\frac{1}{2}}-\text{C}'-\text{D}'x^{\frac{1}{2}}+\cdots \\ &+a\text{A}x^{-\frac{3}{2}}+a\text{B}x^{-1}+a\text{C}x^{-\frac{1}{2}}+a\text{D}+a\text{E}x^{\frac{1}{2}}+\cdots \end{aligned}\right\}=0$$

This gives $a\text{A}-\frac{\text{B}}{4}=0$; $a\text{B}+\text{A}'=0$;

$a\text{C}+2\text{B}'+\frac{3\text{D}}{4}=0$; $a\text{D}+3\text{C}'+2\text{E}=0$;

$a\text{E}+4\text{D}'+\frac{15\text{F}}{4}=0$; $\cdots$ or

$$\text{B}=-\frac{\text{A}'}{a}$$

$$\text{A}=\frac{\text{B}}{4a}=-\frac{\text{A}'}{4a^2}\text{; c is undetermined;}$$

$$\text{D}=-\frac{8\text{B}'}{1.3}-\frac{4a\text{C}}{1.3}=+\frac{32a\text{A}'}{1^2.3^2}-\frac{4a\text{C}}{1.3}$$

$$\text{E}=-\frac{12\text{C}'}{2.4}-\frac{4a\text{D}}{2.4}=-\frac{12.16a^2\text{A}'}{1.3.2^2.4^2}-\frac{4a\text{D}}{2.4}$$

$$\text{F}=-\frac{16\text{D}'}{3.5}-\frac{4a\text{E}}{3.5}=+\frac{16.64a^3\text{A}'}{1.3.2.4.3^2.5^2}-\frac{4a\text{E}}{3.5}$$

$\cdots=\cdots$

We may suppose c either to remain undetermined or to $=0$; in the former case $y=p+qlx$; and in the latter, $y=p+ql.cx$; and in either case there will result the same value of y containing two arbitrary constants.

Ex. 3. $d^2y+ayx^{-\frac{5}{2}}dx=0$.

Here the series which does not fail is the first.

Assume then $q=\text{A}\left\{1-\frac{2^2a}{1.3}x^{-\frac{1}{2}}+\frac{2^4a^2}{1.3.2.4}x^{-1}-\cdots\right\}$

$$=\text{A}+\text{B}x^{-\frac{1}{2}}+\text{C}x^{-1}+\cdots$$

In order to integrate (2), which is

$$\frac{d^2p}{dx^2}+\frac{2}{x}\frac{dq}{dx}-\frac{q}{x^2}+\frac{ap}{x^{\frac{5}{2}}}=0;$$

assume $p = \text{A}'x + \text{B}'x^{\frac{1}{2}} + \text{C}' + \text{D}'x^{-\frac{1}{2}} + \cdots$ then

$$\left.\begin{aligned} &-\frac{\text{B}'}{4}x^{-\frac{3}{2}} + 0.x^{-2} + \frac{3\text{D}'}{4}x^{-\frac{5}{2}} + 2\text{F}'x^{-3} + \cdots \\ &\qquad - 2\left(\frac{\text{B}}{2}x^{-\frac{5}{2}} + \text{C}x^{-3} + \cdots\right) \\ &\qquad - \text{A}x^{-2} - \text{B}x^{-\frac{5}{2}} - \text{C}x^{-3} - \cdots \\ &+ a\text{A}'x^{-\frac{3}{2}} + a\text{B}'x^{-2} + a\text{C}'x^{-\frac{5}{2}} + a\text{D}'x^{-3} + \cdots \end{aligned}\right\};$$

which gives

$$a\text{A}' - \frac{\text{B}'}{4} = 0;\ a\text{B}' - \text{A} = 0;\ a\text{C}' - 2\text{B} + \frac{3\text{D}'}{4} = 0;$$

$$a\text{D}' - 3\text{C} + 2\text{F}' = 0;\ \cdots$$

Suppose $\text{C}' = 0$; then $\text{B}' = \frac{\text{A}}{a}$; $\text{A}' = \frac{\text{B}'}{4a} = \frac{\text{A}}{4a^2}$;

$\text{D}' = \frac{2.4\text{B}}{1.3} = -\frac{8.4a\text{A}}{1^2.3^2}\cdots$; where $y = p + ql.cx$.

There is another transformation of y, by means of which these equations may be integrated. We shall make use of it in the first instance to demonstrate a proposition which confirms in a remarkable manner the solution of Riccati's equation, ch. 4, 19.

12. *$d^2y + ax^n y dx^2 = 0$ is integrable in finite terms if* $n = -\frac{4i}{2i \pm 1}$.

Substitute $y = e^{\int p dx} q$ where p and q are indeterminate functions of x; then the proposed becomes
$d^2q + (2pdq + qdp)dx + (p^2 + ax^n)qdx^2 = 0$.

Assume $p^2 + ax^n = 0$, or $p = x^{\frac{n}{2}}\sqrt{-a} = bx^m$ by substitution; then y, which $= e^{\int p dx} q$, $= e^{\frac{bx^{m+1}}{m+1}} q$.

Also, $d^2q + (2pdq + qdp)dx = 0$, or
$d^2q + 2bx^m dqdx + mbx^{m-1}qdx^2 = 0$.

To integrate this in a series; since x and dx are of 0 dimensions in the first term, and of $m+1$ in the second and third, assume $q = \text{A}x^\alpha + \text{B}x^{\alpha+m+1} + \text{C}x^{\alpha+2m+2} + \cdots$ and there will result

$$q = \text{A}\left\{1 - \frac{bx^{m+1}}{m+1} + \frac{(3m+2)b^2x^{2m+2}}{(m+1)(2m+1)(2m+2)} \cdots\right.$$
$$\left. - \frac{(3m+2)(5m+4)b^3x^{3m+3}}{(m+1)(2m+1)(2m+2)(3m+2)(3m+3)} + \cdots\right\}$$
$$+ \text{A}'\left\{x - \frac{bx^{m+2}}{m+2} + \frac{(3m+4)b^2x^{2m+3}}{(m+2)(2m+2)(2m+3)} \cdots\right.$$
$$\left. - \frac{(3m+4)(5m+6)b^3x^{3m+4}}{(m+2)(2m+2)(2m+3)(3m+3)(3m+4)} + \cdots\right\}.$$

Now one of these series will always terminate when $m = -\frac{2i}{2i \pm 1}$, *i. e.* when $n = -\frac{4i}{2i \pm 1}$; and consequently a conjugate value of q or of y may be obtained in finite terms; and having obtained this, the complete value of y may be found as in ch. 6. 28. cor. 2.

The proposed form is reducible to Riccati's by substituting $y = e^{\int z dx}$, when it becomes $dz + z^2dx = -ax^ndx$.

Euler has integrated the proposed by changing its form into $d^2y - b^2yx^ndx = 0$, and developing q in a descending series, as in the next article.

13. *Required to integrate* $d^2y - b^2yx^ndx = 0$ *when* $n = -\frac{4i}{2i \pm 1}$.

By the last article, if we substitute $y = e^{bt}q$, where $t = \frac{x^{m+1}}{m+1}$ and $2m = n$, the integration of the proposed is reduced to that of $d^2q + 2bx^mdqdx + mbx^{m-1}qdx^2 = 0$.

To develope q in a *descending* series, assume $q = \text{A}x^\alpha + \text{B}x^{\alpha-\beta} + \text{C}x^{\alpha-2\beta} + \cdots$; then, arranging the terms in a reversed order,

$$mb\text{A}x^{\alpha+m-1} + mb\text{B}x^{\alpha-\beta+m-1} + mb\text{C}x^{\alpha-2\beta+m-1} + \cdots$$
$$+2b(\alpha\text{A} + (\alpha-\beta)\text{B} + (\alpha-2\beta)\text{C} + \cdots)$$
$$+\alpha(\alpha-1)\text{A} + (\alpha-\beta)(\alpha-\beta-1)\text{B} + \cdots$$
$$= 0.$$

Suppose A to be undetermined; then $2b\alpha + mb = 0$, or $\alpha = -\frac{m}{2}$; $\alpha - \beta + m - 1 = \alpha - 2$, or $\beta = 1 + m = 1 - 2\alpha$; and

the developement of q is of the form

$q = \text{A}x^{\alpha} + \text{B}x^{3\alpha-1} + \text{C}x^{5\alpha-2} + \text{D}x^{7\alpha-3} + \cdots$

To determine B, C, D, $\cdots$ we have

$-2b\beta\text{B} + \alpha(\alpha-1)\text{A} = 0$; $-4b\beta\text{C} + (\alpha-\beta)(\alpha-\beta-1)\text{B} = 0$; ...

or $\text{B} = \dfrac{\alpha(\alpha-1)}{2\beta b}\text{A}$; $\text{C} = \dfrac{(3\alpha-1)(3\alpha-2)}{4\beta b}\text{B}$;

$\text{D} = \dfrac{(5\alpha-2)(5\alpha-3)}{6\beta b}\text{C}$; $\text{E} = \dfrac{(7\alpha-3)(7\alpha-4)}{8\beta b}$; $\cdots$ a series

which will always terminate when $(2i \pm 1)\alpha - i = 0$, or

$\alpha = \dfrac{i}{2i\pm1}$, *i. e.* when n, which $= 2m = -4\alpha$,

$= -\dfrac{4i}{2i\pm1}$.

These being substituted, there will result

$$q = \text{A}\left\{ x^{\alpha} - \frac{\alpha(\alpha-1)}{2(2\alpha-1)b}x^{3\alpha-1} \cdots \right.$$
$$\left. + \frac{\alpha(\alpha-1)(3\alpha-1)(3\alpha-2)}{8(2\alpha-1)^2b^2}x^{5\alpha-2} - \cdots \right\}.$$

But we have only one arbitrary constant, viz. A; and to introduce another, we observe that the proposed may be integrated on the supposition either of b positive or of b negative.

Let A, as before, correspond to $+b$; A′, to $-b$; then, when b is negative,

$$q = \text{A}'\left\{ x^{\alpha} + \frac{\alpha(\alpha-1)}{2(2\alpha-1)b}x^{3\alpha-1} \cdots \right.$$
$$\left. + \frac{\alpha(\alpha-1)(3\alpha-1)(3\alpha-2)}{8(2\alpha-1)^2b^2}x^{5\alpha-2} + \cdots \right\}.$$

Let $\left.\begin{matrix}\text{A.P} = \text{sum of all the terms in which } b\text{'s exponent} \\ \text{is even}\end{matrix}\right\}$;

$\left.\begin{matrix}\text{A.}b\text{Q} = \text{sum of all the terms in which } b\text{'s exponent} \\ \text{is odd}\end{matrix}\right\}$

then the conjugate values of q are $q = \text{A}(\text{P} + b\text{Q})$ and $q' = \text{A}'(\text{P} - b\text{Q})$; and the complete primitive is

$y = \text{P}(\text{A}e^{bt} + \text{A}'e^{-bt}) + b\text{Q}(\text{A}e^{bt} - \text{A}'e^{-bt})$ where $t = \dfrac{x^{m+1}}{m+1}$.

Cor. 1. When a in the last article is positive, b contains $\sqrt{-1}$, and y may be expressed as a circular function of x.

For let $b = \beta\sqrt{-1}$; then (vol. i. ch. 2, 36)

$e^{\beta t\sqrt{-1}} = \cos.\beta t + \sqrt{-1}\sin.\beta t$ } $\therefore$

$e^{-\beta t\sqrt{-1}} = \cos.\beta t - \sqrt{-1}\sin.\beta t$ }

$y = \text{P}\{(\text{A} + \text{A}')\cos\beta t + (\text{A} - \text{A}')\sqrt{-1}\sin\beta t\}$ . . . .

$+ b\text{Q}\{(\text{A} - \text{A}')\cos.\beta t + (\text{A} + \text{A}')\sqrt{-1}\sin.\beta t\}$

$= \text{P}\{c\cos.\beta t + c'\sin\beta t\} + \beta\text{Q}\{c'\cos.\beta t - c\sin.\beta t\}$,

where $c = \text{A} + \text{A}'$ and $c' = (\text{A} - \text{A}')\sqrt{-1}$ . . . . .

$= \text{C}\text{P}(\sin.\beta t + \text{C}') + \text{C}\beta\text{Q}(\cos.\beta t + \text{C}')$,

where $\text{A} + \text{A}' = \text{C}\sin.\text{C}'$ and $\text{A} - \text{A}'\sqrt{-1} = \text{C}\cos.\text{C}'$.

Cor. 2. Riccati's form is integrable by this method.

For let the form be $dz + z^2dx = ax^ndx$; then substituting $z = \dfrac{dy}{ydx}$ and differentiating, it becomes

$d^2y - ayx^ndx^2 = 0$.

Ex. 1. $d^2y - b^2ydx^2 = 0$.

Here $n = 0 \therefore \alpha = 0 = m \therefore t = x$; and $q = \text{A}$, $q' = \text{A}'$

$\therefore y = \text{A}e^{bx} + \text{A}'e^{-bx}$.

If $b^2 = -a^2$, $y = c\cos.ax + c'\sin.ax = \text{C}\sin.(ax + \text{C}')$.

Ex. 2. $d^2y - b^2yx^{-4}dx^2 = 0$.

Here $m = -2$; $t = -\dfrac{1}{x}$ and $\alpha = 1 \therefore q = \text{A}x$ and

$y = e^{-\frac{b}{x}}\text{A}x + e^{\frac{b}{x}}\text{A}'x = x(\text{A}'e^{\frac{b}{x}} + \text{A}e^{-\frac{b}{x}})$.

If $b^2 = -a^2$, $y = \text{C}x\sin.(ax^{-1} + \text{C}')$.

Ex. 3. $d^2y - b^2yx^{-\frac{4}{3}}dx^2 = 0$.

Here $m = -\frac{2}{3}$; $t = 3x^{\frac{1}{3}}$ and $\alpha = \frac{1}{3} \therefore$

$q = \text{A}\left\{x^{\frac{1}{3}} - \dfrac{1}{3b}\right\}$ and $q' = \text{A}'\left\{x^{\frac{1}{3}} + \dfrac{1}{3b}\right\}$ and

$y = x^{\frac{1}{3}}(\text{A}e^{bt} + \text{A}'e^{-bt}) - \dfrac{1}{3b}(\text{A}e^{bt} - \text{A}'e^{-bt})$, where $t = 3x^{\frac{1}{3}}$.

If $b^2 = -a^2$, $y = x\left\{(\text{A}+\text{A}')\cos.at + (\text{A}-\text{A}')\sqrt{-1}\sin.at\right\}$

$- \dfrac{1}{3a\sqrt{-1}}\left\{(\text{A} - \text{A}')\cos.at + (\text{A} + \text{A}')\sqrt{-1}\sin.at\right\}$ . . .

$= x^{\frac{1}{3}}(c\cos.at + c'\sin.at) + \dfrac{1}{3a}(c'\cos.at - c\sin.at)$

$= \text{C}x^{\frac{1}{3}}\sin.(3ax^{\frac{1}{3}} + \text{C}') + \dfrac{\text{C}}{3a}\cos.(3ax^{\frac{1}{3}} + \text{C}')$.

Ex. 4. $d^2y - b^2yx^{-\frac{8}{5}}dx^2 = 0.$

Here $m = -\frac{4}{5}$; $t = 5x^{\frac{1}{5}}$ and $\alpha = \frac{2}{5}$ $\therefore$

$$q = \text{A}\left\{ x^{\frac{2}{5}} - \frac{3x^{\frac{1}{5}}}{5b} + \frac{3}{5^2b^2} \right\} \therefore y = \left\{ x^{\frac{2}{5}} + \frac{3}{5^2b^2} \right\} (\text{A}e^{bt} + \text{A}'e^{-bt}$$

$$- \frac{3x^{\frac{1}{5}}}{5b} (\text{A}e^{bt} - \text{A}'e^{-bt}), \text{ where } t = 5x^{\frac{1}{5}}.$$

If $b^2 = -a^2$, $y = \text{c}\left\{ x^{\frac{2}{5}} - \frac{3}{5^2a^2} \right\} \text{sin.}(5ax^{\frac{1}{5}} + \text{c}') \ . \ . \ .$

$$+ \frac{3\text{c}x^{\frac{1}{5}}}{5a} \text{cos.}(5ax^{\frac{1}{5}} + \text{c}').$$

Ex. 5. $d^2y + ayx^{-\frac{8}{3}}dx^2 = 0.$

To integrate this without referring to any formula, substitute $y = e^{\int pdx}q$; then it becomes

$$d^2q + 2pdqdx + qdpdx + (p^2 + ax^{-\frac{8}{3}})qdx^2 = 0.$$

Suppose $p^2 + ax^{-\frac{8}{3}} = 0$, or $p = x^{-\frac{4}{3}}\sqrt{-a} = bx^{-\frac{4}{3}}$,

by substitution; then $\frac{d^2q}{dx^2} + 2p\frac{dq}{dx} + q\frac{dp}{dx} = 0$, or

$$\frac{d^2q}{dx^2} + 2bx^{-\frac{4}{3}}\frac{dq}{dx} - \frac{4bx^{-\frac{7}{3}}q}{3} = 0.$$

Assume $q = \text{A}x^{\alpha} + \text{B}x^{\alpha-\beta} + \text{c}x^{\alpha-2\beta} + \cdots$ then, reversing the order,

$$- \frac{4b\text{A}}{3}x^{\alpha-\frac{7}{3}} - \frac{4b\text{B}}{3}x^{\alpha-\beta-\frac{7}{3}} - \frac{4b\text{c}}{3}x^{\alpha-2\beta-\frac{7}{3}} - \ldots$$

$$+ 2b\alpha\text{A} \quad + 2b(\alpha-\beta)\text{B} \quad + 2b(\alpha-2\beta)\text{c} \quad + \ldots$$

$$+ \alpha(\alpha-1)\text{A} + (\alpha-\beta)(\alpha-\beta-1)\text{B} \quad + \ldots$$

$$= 0.$$

To satisfy this, we have A undetermined; $\alpha = \frac{2}{3}$; $\beta = -\frac{1}{3}$.

Also $\text{B} = \frac{\text{A}}{3b}$ and $\text{c} = \text{D} = \ldots = 0$; wherefore

$$q = \text{A}\left\{ x^{\frac{2}{3}} + \frac{x}{3b} \right\}.$$

This furnishes only a conjugate value; but since $b^2 = -a$,

we have $b = \pm\sqrt{-a}$, and consequently another value is $q' = A'\left\{x^{\frac{2}{3}} - \frac{x}{3b}\right\}$.

Now $p = \pm bx^{-\frac{4}{3}} \therefore \int p dx = \mp 3bx^{-\frac{1}{3}} =$ by substitution $\mp bt$; then $y = x^{\frac{2}{3}}(Ae^{-bt} + A'e^{bt}) + \frac{x}{3b}(Ae^{-bt} - A'e^{bt})$,

where $b = \sqrt{-a}$ and $t = 3x^{-\frac{1}{3}}$.

When a is positive, this is under an irrational form; and may be expressed as a circular function, as in the former examples.

We shall give only one more of Euler's examples, which is the developement of the form of the article in a series in terms of i.

Ex. 6. $d^2y - b^2x^n y dx^2 = 0$ where $n = -\frac{4i}{2i-1}$.

Substitute $y = e^{bt}q$ where $t = -\frac{1}{2i-1}x^{-\frac{1}{2i-1}}$, or

$\frac{dt}{dx} = x^{-\frac{2i}{2i-1}}$; then

$$\frac{d^2q}{dx^2} + 2bx^{-\frac{2i}{2i-1}}\frac{dq}{dx} - \frac{2bix}{2i-1}^{-\frac{4i-1}{2i-1}}q = 0.$$

Assume $q = Ax^{\frac{i}{2i-1}} + Bx^{\frac{i+1}{2i-1}} + Cx^{\frac{i+2}{2i-1}} + \ldots$; then

$$\frac{dq}{dx} = \frac{i}{2i-1}Ax^{-\frac{i-1}{2i-1}} + \frac{i+1}{2i-1}Bx^{-\frac{i+2}{2i-1}} + \frac{i+2}{2i-1}Cx^{-\frac{i-3}{2i-1}} \ldots \therefore$$

$$-\frac{2bi}{2i-1}Ax^{-\frac{3i-1}{2i-1}} - \frac{2bi}{2i-1}Bx^{-\frac{3i-2}{2i-1}} - \frac{2bi}{2i-1}Cx^{-\frac{3i-3}{2i-1}} \ldots$$

$$+\frac{2bi}{2i-1}A + \frac{2b(i+1)}{2i-1}B + \frac{2b(i+2)}{2i-1}C \ldots$$

$$-\frac{i(i-1)}{(2i-1)^2}A - \frac{(i+1)(i-2)}{(2i-1)^2}B \ldots$$

$= 0$.

Whence $-\frac{i(i-1)A}{(2i-1)^2} + \frac{2bB}{2i-1} = 0$;

$-\frac{(i+1)(i-2)B}{(2i-1)^2} + \frac{4bC}{2i-1} = 0$; ... or

$$\text{B} = \frac{i(i-1)}{2(2i-1)b}\text{A}\,;\ \text{C} = \frac{(i+1)(i-2)}{4(2i-1)b}\text{B}\,;\ \text{D} = \frac{(i+2)(i-3)}{6(2i-1)b}\text{C}\,;\&c.$$

$$\therefore q = \text{A}t^{-i}\left\{1 - \frac{i(i-1)}{2bt} + \frac{i(i^2-1)(i-2)}{2.4b^2t^2} \quad \&c. \quad - \frac{i(i^2-1)(i^2-4)(i-3)}{2.4.6b^3t^3} - \&c.\right\} \text{ and}$$

$$y = t^{-1}\left\{1 + \frac{i(i^2-1)(i-2)}{2.4b^2t^2} \quad \&c. \quad + \frac{i(i^2-1)(i^2-4)(i^2-9)(i-4)}{2.4.6.8b^4t^4} + \&c.\right\}(\text{A}e^{bt} + \text{A}'e^{-bt})$$

$$- t^{-1}\left\{\frac{i(i-1)}{2bt} + \frac{i(i^2-11)(i^2-4)(i-3)}{2.4.6b^3t^3} + \&c.\right\}(\text{A}e^{bt} - \text{A}'e^{-bt}).$$

If $n = -\frac{4i}{2i+1}$, the same series obtains, making i negative; and $t = (2i+1)x^{\frac{i}{2i+1}}$.

It may be shown, as in the former examples, that the primitive of $d^2y + a^2yx^{\frac{4i}{2i-1}}dx^2 = 0$ is

$$y = ct^{-1}\left\{1 - \frac{i(i^2-1)(i-2)}{2.4a^2t^2} \quad \&c. \quad + \frac{i(i^2-1)(i^2-4)(i^2-9)(i-4)}{2.4.6.8a^4t^4} - \&c.\right\}\sin.(at + c')$$

$$+ c't^{-1}\left\{\frac{i(i-1)}{2at} - \frac{i(i^2-1)(i^2-4)(i-3)}{2.4.6a^3t^3} + \&c.\right\}\cos.(at + c).$$

The same series will give the primitive of $d^2y + a^2yx^{\frac{4i}{2i+1}}dx^2 = 0$; if $t = \frac{1}{2i+1}x^{\frac{1}{2i+1}}$, and i be made negative.

Euler has integrated by this method other forms which are more complicated; but he observes, that it is a very inconvenient method unless the example is linear, *i. e.* of the form $d^2y + \text{M}dx^2dy + \text{N}ydx^2 = \text{X}dx^2$.

The following form, of which we shall give but a brief perfect account, he has fully discussed.

14. *Required to integrate in a series*
$x^2(a + bx^n)d^2y + x(c + ex^n)dxdy + (f + gx^n)ydx^2 = 0.$

Substituting $z = x^n$, we have

$z^{\frac{2}{n}}(a+bz)d^2y+z^{\frac{1}{n}}(c+ez)z^{\frac{1}{n}-1}\frac{dzdy}{n}+(f+gz)z^{\frac{2}{n}-2}y\frac{dz^2}{n^2}=0,$

or $z^2(a+bz)d^2y+z\left(\frac{c}{n}+\frac{ez}{n}\right)dzd+\left(\frac{f}{n^2}+\frac{gz}{n^2}\right)ydz^2=0$; so that we need consider only the form

$x^2(a+bx)d^2y+x(c+ex)dxdy+(f+gx)ydx^2=0.$

Assume $y=\text{A}x^\alpha+\text{B}x^{\alpha+1}+\text{C}x^{\alpha+2}+\ldots$ then

$$\left.\begin{array}{llllll} a(a-1)a\text{A}x^\alpha+(a+1)a\text{B}x^{\alpha+1}+(a+2)(a+1)a\text{C}x^{\alpha+2}+\ldots \\ +\quad a\text{C}\text{A} & +\ (a+1)c\text{B} & +\quad (a+2)c\text{C} & +\ldots \\ +\quad f\text{A} & +\quad f\text{B} & +\quad f\text{C} & +\ldots \\ & +a(a-1)b\text{A} & +\quad (a+1)a\text{B} & +\ldots \\ & +\quad ae\text{A} & +\quad (a+1)e\text{B} & +\ldots \\ & +\quad g\text{A} & +\quad g\text{B} & +\ldots \end{array}\right\} = 0$$

This is satisfied by supposing A indeterminate, and the value of a obtained from the quadratick equation

$a(a-1)a+ac+f=0$ (1).

Let the roots of (1) be a and a'; and let A and A′ be the corresponding values of A: then, by means of this series, two conjugate values of y, and consequently its complete value, may be found as before.

The equations for determining B, C, . . . are

$\{(a+1)aa+(a+1)c+f\}\text{B}+\{a(a-1)b+ae+g\}\text{A}=0$

$\{(a+2)(a+1)a+(a+2)c+f\}\text{C}\ .\ .\ .\ .\ .\ .\ .\ .\ .$

$+\{(a+1)ab+(a+1)e+g\}\text{B}=0$

$\{(a+3)(a+2)a+(a+3)c+f\}\text{D}\ .\ .\ .\ .\ .\ .\ .\ .\ .$

$+\{(a+2)(a+1)b+(a+2)e+g\}\text{C}=0$

. . .; which will terminate when

$(a+i)(a+i-1)b+(a+i)e+g=0.$

If a descending series be assumed for y,

$y=\text{A}x^\alpha+\text{B}x^{\alpha-1}+\ldots$, the equation for determining a is

$a(a-1)b+ae+g=0$ (2).

15. There are two cases in which the first solution gives only a particular primitive, viz. when $a=0$; and when $4af=(a-c)^2$, in which latter case the two series become equal.

The second solution fails when $b=0$ and when $4bg=(b-e)^2$. If both solutions fail, or if the difference between a and a' is divisible by n, Euler substitutes $y=p+ql.cx$.

If, when $a = 0$, we have also $c = 0$, the first solution fails to give even a particular primitive.

Ex. 1. $xd^2y + dxdy + gx^{n-1}ydx^2 = 0$, or
$x^2d^2y + xdxdy + gx^nydx^2 = 0$.

Here $a = 1$; $b = 0$; $c = 1$; $e = 0$ and $f = 0$: $\therefore a = a'$.

Substitute $y = p + ql.cx$; then
$x^2d^2p + xdxdp + 2xdxdq + gx^npdx^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad .$
$+ (x^2d^2q + xdxdq + gx^nqdx^2)l.cx = 0$; which divides into
$x^2d^2p + xdxdp + 2xdxdq + gx^npdx^2 = 0$ (1)
$x^2d^2q + xdxdq + gx^nqdx^2 = 0$ (2).

Assume $q = A + Bx^n + Cx^{2n} + Dx^{3n} + \cdots$
$p = A' + B'x^n + C'x^{2n} + D'x^{3n} + \cdots$; and from

(2) it will result that $B = -\frac{gA}{n^2}$; $C = -\frac{gB}{4n^2} = +\frac{g^2A}{1.4n^4}$;

$D = -\frac{gC}{9n^2} = -\frac{g^3A}{1.4.9n^6}$; $\cdots$

To integrate (1), we may assume $A' = 0$, because it is included in cq and we have $B' = +\frac{2gA}{n^3}$; $C' = -\frac{6g^2A}{1.8n^5}$;

$D' = +\frac{22g^3A}{1.8.27n^7} \cdots$ or

$$y = A\left\{\frac{2g}{n^3}x^n - \frac{6g^2}{1.8n^5}x^{2n} + \frac{22g^3}{1.8.27n^7}x^{3n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \right.$$
$$\left. - \frac{100g^4}{1.8.27.64n^9}x^{4n} + \cdots\right\}$$
$$+ Al.cx\left\{1 - \frac{g}{n^2}x^n + \frac{g^2}{1.4n^4}x^{2n} - \frac{g^3}{1.4.9n^6}x^{3n} \quad . \quad . \quad . \quad . \quad . \right.$$
$$\left. + \frac{g^4}{1.4.9.16n^8}x^{4n} - \cdots\right\}.$$

Ex. 2. $x(1 - x^2)d^2y - (1 + x^2)dxdy + xydx^2 = 0$.

Substitute $y = p + ql.cx$; then

$$\left.\begin{aligned}\frac{dy}{dx} &= \frac{dp}{dx} + l.cx\frac{dq}{dx} + \frac{q}{x}\\ \frac{d^2y}{dx^2} &= \frac{d^2p}{dx^2} + l.cx\frac{d^2q}{dx^2} + \frac{2dq}{xdx} - \frac{q}{x^2}\end{aligned}\right\}\cdots$$

$$x(1-x^2)\frac{d^2p}{dx^2} - (1+x^2)\frac{dp}{dx} + xp \Big\} = 0 \ (1); \text{ and}$$
$$+ 2(1-x^2)\frac{dq}{dx} - \frac{2q}{x} \Big\}$$

$$x(1-x^2)\frac{d^2q}{dx^2} - (1+x^2)\frac{dq}{dx} + xq = 0 \ (2).$$

To integrate (2), when reduced to the form of the article, it is $x^2(1-x^2)d^2q - x(1+x^2)dqdx + x^2qdx^2 = 0$.

Since n here $= 2$, assume

$q = \text{A}x^\alpha + \text{B}x^{\alpha+2} + \text{C}x^{\alpha+4} + \dots$ then

$$\left.\begin{array}{l}
\alpha(\alpha-1)\text{A}x^\alpha \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
+(\alpha+2)(\alpha+1)\text{B}x^{\alpha+2}+(\alpha+4)(\alpha+3)\text{C}x^{\alpha+4}+\dots \\
- \quad \alpha(\alpha-1)\text{A}x^{\alpha+2}-(\alpha+2)(\alpha+1)\text{B}x^{\alpha+4}-\dots \\
- \quad \alpha\text{A}x^\alpha \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
- \quad (\alpha+2)\text{B}x^{\alpha+2}- \quad (\alpha+4)\text{C}x^{\alpha+4}-\dots \\
- \quad \alpha\text{A}x^{\alpha+2}- \quad (\alpha+2)\text{B}x^{\alpha+4}-\dots \\
+ \quad \text{A}x^{\alpha+2}+ \quad \text{B}x^{\alpha+4}-\dots
\end{array}\right\} = 0.$$

This gives $(\alpha^2 - 2\alpha)\text{A} = 0$; or A is indeterminate and $\alpha = 0$, $\alpha' = 2$.

Assume then $q = \text{A}x^2 + \text{B}x^4 + \text{C}x^6 + \dots$

$$p = \text{A}' + \text{B}'x^2 + \text{C}'x^4 + \dots$$

$2.4\text{B} - 1.3\text{A} = 0$; $4.6\text{C} = 3.5\text{B}$; ... or $\text{B} = \frac{1.3}{2.4}\text{A}$;

$$\text{C} = \frac{1.3^2.5}{2.4^2.6}\text{B}; \ \text{D} = \frac{1.3^2.5^2.7}{2.4^2.6^2.8}\text{C}; \dots$$

To determine p, we have from (1)

$$\left.\begin{array}{llll}
2\text{B}'x & + 4.3\text{C}'x^3 & + \dots \\
 & - 2\text{B}' & - \dots \\
-2\text{B}' & - 4\text{C}' & - \dots \\
 & - 2\text{B}' & - \dots \\
+ \text{A}' & + \text{B}' & + \dots \\
+4\text{A} & + 8\text{B} & + \dots \\
 & - 4\text{A} & - \dots \\
-2\text{A} & - 2\text{B} & - \dots
\end{array}\right\} = 0; \text{ which gives the following:}$$

$$\text{A}' = -2\text{A}; \ \text{B}' = 0; \ \text{C}' = \frac{1.3}{2.4}\text{B}' + \frac{2.7}{2^2.4^2}\text{A}; \ . \ . \ . \ . \ . \ .$$

$$\text{D}' = \frac{3.5}{4.6}\text{C}' + \frac{2.21}{4^2.6^2}\text{B};$$

$$E' = \frac{5.7}{6.8}D' + \frac{2.43}{6^2.8^2}C \cdots$$

Having thus determined the values of p and q, they are to be substituted in $y = p + ql.cx$.

Euler has shown that this method will always succeed when the difference between the values α and α' is divisible by n.

Cor. If the roots of the quadratick contain $\sqrt{-1}$, let $\alpha = \beta \pm \gamma\sqrt{-1}$; then, substituting for $x^{\alpha+in}$, which $= x^{\beta+in\pm\gamma\sqrt{-1}}$, the expression $x^{\beta+in}(\cos.\gamma lx \pm \sqrt{-1}\sin.\gamma lx)$, the $\sqrt{-1}$ will disappear.

If this fails to give a complete primitive, we may substitute $y = p\cos.klx + q\sin.klx$; and deducing the values of $\frac{dy}{dx}$ and of $\frac{d^2y}{dx^2}$, and substituting these in the proposed, it will divide into two equations, if we make the coefficients of $\cos.klx$ and of $\sin.klx$ each $= 0$.

16. When the law of the exponents cannot be ascertained from any arrangement of the terms, which is the case when the equation is of a higher degree than the first, recourse must be had to Newton's method of successive substitutions.

To introduce an arbitrary constant, let a and b be two corresponding values of the variables; then by Euler's Theorem (vol. i. ch. 4. 45.)

$$y = b + (p)(x-a) + (q)\frac{(x-a)^2}{1.2} + \ldots \text{ where } (p), (q), \ldots$$

denote the fluxional coefficients corresponding to $x = a$ or $y = b$.

Let $x - a = h$; then $y = b + (p)\frac{h}{1} + (q)\frac{p^2}{1.2} + \ldots$;

and if h be supposed small, the first approximate value is

$$y = b + (p)\frac{h}{1}.$$

Let this $= b'$, the corresponding value of a being a' or $a + h$. Considering these as new values of y and x, a nearer approximation may be made; and by repeating the process, we find a value of y corresponding to any value of x.

If the equation is of the second order, let it be under the form $q = f(x, y, p)$; then, in order to find the complete primitive, we must not only know corresponding values of x and y as before, viz. $x = a$, $y = b$; but also another condition must be added, viz. that the corresponding value of p is $p = c$.

This agrees with what has already appeared, that when the solution of a problem depends upon an equation of the second order, we must have two conditions in order to determine it: 1st, that when $x = a$, $y = b$; and 2dly, that when $x = a$, $p = c$.

Generally, if the equation is of the nth order, the $n - 1$ first fluxional coefficients must be considered as arbitrary constants.

This method of approximation, which is Euler's, is inconvenient in practice, not only on account of the length of the calculations, but also because the errors of each operation influence the succeeding, which is not the case in the method explained ch. 2, art. 3, where the operations are independent of each other. Lagrange therefore proposes to develope y in a continued fraction, as in art. 4.

When this can be accomplished, an approximation is obtained which never fails, and which in general is sufficiently rapid; the equations, however, for determining Ax^{α}, Bx^{β}, Cx^{γ} . . ., after the two or three first approximations, become so complicated, that we shall subjoin but few examples.

Cor. This method, it is obvious, furnishes the means of representing any proposed function in a continued fraction; for we have only to differentiate the function, and then integrate the resulting equation in a continued fraction.

17. EXAMPLES.

Ex. 1. $mydx + (1 + x)dy = 0$.

Let the first value of y be $y = Ax^{\alpha}$; then, by substitution, $(m + \alpha)Ax^{\alpha} + \alpha Ax^{\alpha-1} = 0$, or $(m + \alpha)Ax + \alpha A = 0$, which, when $x = \frac{1}{\infty}$, becomes $\alpha A = 0$; which may be satisfied by $\alpha = 0$, A remaining indeterminate.

Let $y = \frac{A}{1 + Bx^{\beta}}$; then, by substitution,

$m(1 + Bx^{\beta}) - (1 + x)\beta Bx^{\beta-1} = 0$, or

$m - \beta \text{B} x^{\beta-1} + (m-\beta)\text{B}x^{\beta} = 0$, which is satisfied by $\beta = 1$, $\text{B} = m$.

Next let $y = \dfrac{\text{A}}{1 + \dfrac{mx}{1+\text{C}x^{\gamma}}}$; then, by substitution, it will appear that $\gamma = 1$, $\text{C} = -\dfrac{m-1}{2}$; and by repeating the process, we shall find

$$y = \cfrac{\text{A}}{1+mx}$$

$$\cfrac{}{1 - \cfrac{m-1}{1} \cdot \cfrac{x}{2}}$$

$$\cfrac{}{1 + \cfrac{m+1}{3} \cdot \cfrac{x}{2}}$$

$$\cfrac{}{1 - \cfrac{m-2}{3} \cdot \cfrac{x}{2}}$$

$$\cfrac{}{1 + \cfrac{m+2}{5} \cdot \cfrac{x}{2}}$$

$$\cfrac{}{1 - \cfrac{m-3}{5} \cdot \cfrac{x}{2}}$$

$$\cfrac{}{1 + \ldots}$$

Integrating the example, $(1+x)^m y = \text{A}$, or

$$(1+x)^m = \frac{\text{A}}{y} = 1 + \cfrac{mx}{1 - \cfrac{m-1}{1} \cdot \cfrac{x}{2}}$$

$$\cfrac{}{1 + \ldots}$$

Ex. 2. $dx - (1+x^n)dy = 0$, or $(1+x^n)\dfrac{dy}{dx} - 1 = 0$.

Let $y = \text{A}x^{\alpha}$; then $(1+x^n)\alpha\text{A}x^{\alpha-1} = 0 \therefore \alpha = 1$, $\text{A} = 1$.

Let $y = \dfrac{x}{1+\text{B}x^{\beta}}$; then

$(1 + x^n)(1 + (1 - \beta)\text{B}x^\beta) - (1 + \text{B}x^\beta)^2 = 0$, or
$-(1 + \beta)\text{B}x^\beta - \text{B}^2x^{2\beta} + (1 - \beta)\text{B}x^{\beta+n} + x^n = 0 \therefore$
$\beta = n$, $\text{B} = \dfrac{1}{1+n}$; and repeating the process,

$$y = \cfrac{x}{1 + \cfrac{x^n}{n+1}}$$

$$\cfrac{}{1 + \cfrac{n^2x^n}{(n+1)(2n+1)}}$$

$$\cfrac{}{1 + \cfrac{(n+1)^2x^n}{(2n+1)(3n+1)}}$$

$$\cfrac{}{1 + \cfrac{(2n)^2x^n}{(3n+1)(4n+1)}}$$

$$\cfrac{}{1 + \cfrac{(2n+1)^2x^n}{(4n+1)(5n+1)}}$$

$$\cfrac{}{1 + \cdots}$$

Cor. Since $y = \int \dfrac{dx}{1+x^n}$, we have,

if $n = 1$, $l(1 + x) = \cfrac{x}{1 + \cfrac{x}{1.2}}$

$$\cfrac{}{1 + \cfrac{x}{3.2}}$$

$$\cfrac{}{1 + \cfrac{2x}{3.2}}$$

$$\cfrac{}{1 + \cfrac{2x}{5.2}}$$

$$\cfrac{}{1 + \cfrac{3x}{5.2}}$$

$$\cfrac{}{1 + \ldots}$$

$$\text{if } n = 2,\ \tan.^{-1}x = \cfrac{x}{1 + \cfrac{x^2}{1.3}}$$

$$\cfrac{}{1 + \cfrac{4x^2}{3.5}}$$

$$\cfrac{}{1 + \cfrac{9x^2}{5.7}}$$

$$\cfrac{}{1 + \cfrac{16x^2}{7.9}}$$

$$\cfrac{}{1 + \cdots}$$

18. There are other methods of integrating by approximation total equations, which we shall entirely omit: one is Lagrange's method of the variation of the parameters, of which we have seen the use in ch. 6, art. 32; another is Newton's method of successive substitution combined with the known processes of integrating equations of the first degree. It is this which is so much used in the solution of the problem of the three bodies; but it cannot be rendered intelligible to a student who is wholly ignorant of the science of Physical Astronomy. Omitting these, as well as the geometrical construction of fluxional equations, which is also a method of approximation, we shall conclude this part of the work with a few examples of integrating *partial* equations by series.

19. Euler observed of several partial equations of the second order which are not integrable, that an infinite number of solutions may be found belonging to them. The following example, for instance, though simple in its form, is unintegrable.

Ex. $\dfrac{d^2z}{dxdy} = az.$

Substituting $z = \text{A}e^{\alpha x}\text{Y}$, where A and α are indeterminate and $\text{Y} = fy$, it becomes $\alpha\dfrac{d\text{Y}}{dy} = a\text{Y}$ or $\dfrac{d\text{Y}}{\text{Y}} = \dfrac{ady}{\alpha}$; whence, integrating, $\text{Y} = e^{\frac{ay}{\alpha}}$ and $z = \text{A}e^{\alpha x + \frac{ay}{\alpha}}$.

Since A and α remain undetermined, it is manifest that

the primitive of the proposed is

$z = \text{A}e^{\alpha x + \frac{ay}{\alpha}} + \text{B}e^{\beta x + \frac{ay}{\beta}} + \text{C}e^{\gamma x + \frac{ay}{\gamma}} + \cdots$ where the series may be continued to any number of terms ad libitum.

Thus, if three terms be taken, and it be differentiated, it will appear that this value of z satisfies the example.

Cor. We may express z as a circular function of x and y by supposing α to contain $\sqrt{-1}$; or thus:

Suppose $-\alpha\beta = a$, which leaves one of the quantities α and β indeterminate; then, it appears from differentiation, that $z = \text{A}\sin.(\alpha x + \beta y)$ and $z = \text{B}\cos.(\alpha x + \beta y)$: each satisfies the example.

Suppose α to have an infinite number of values, α, α', α'' $\cdots$; then the corresponding values of β are

$-\frac{a}{\alpha}, -\frac{a}{\alpha'}, -\frac{a}{\alpha''} - \cdots$ and we have

$$\left.\begin{aligned} z = {} & \text{A}\sin.\left(\alpha x - \frac{ay}{\alpha}\right) + \text{A}'\sin.\left(\alpha' x - \frac{ay}{\alpha'}\right) \cdots \\ & + \text{A}''\sin.\left(\alpha'' x - \frac{ay}{\alpha''}\right) + \cdots \\ & + \text{B}\cos.\left(\alpha x - \frac{ay}{\alpha}\right) + \text{B}'\cos.\left(\alpha' x - \frac{ay}{\alpha'}\right) \cdots \\ & + \text{C}''\cos.\left(\alpha'' x - \frac{ay}{\alpha''}\right) + \cdots \end{aligned}\right\}, \text{ where}$$

A, A′, $\cdots$ B, B′, $\cdots$ and α, α', $\cdots$ are all indeterminate.

20. Partial equations may be integrated in series by means of Maclaurin's theorem.

For by that theorem, $z = \text{z} + \frac{dz}{dx}\frac{x}{1} + \frac{d^2\text{z}}{dx^2}\frac{x^2}{1.2} + \cdots$

where z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, $\cdots$ represent the values of z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$ $\cdots$ respectively when $x = 0$, and consequently are functions which contain only y and constant quantities.

First, let the proposed equation be of the *first* order; and let it be reduced to the form $\frac{dz}{dx} = \text{F}\left(x, y, z, \frac{dz}{dy}\right)$; then by successive differentiation we may obtain the values of $\frac{d^2z}{dx^2}$, $\frac{d^3z}{dx^3}$ $\cdots$; and consequently we may find

$\frac{dz}{dx}, \frac{d^2z}{dx^2}, \frac{d^3z}{dx^3}$ . . . in terms of z, which may be considered as an arbitrary function of y.

If any of the coefficients become infinite when $x = 0$, the failure of the theorem is to be prevented by substituting $x - a$ for x; or, which is the same thing, substituting $x + a$ for x, and continuing to develope the series by the powers of x.

Next, let the proposed be of the *second* order; and let it be under the form $\frac{d^2z}{dx^2} = \text{F}\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dy^2}, \frac{d^2z}{dxdy}\right)$.

Adopting the same notation as before, let $z = \phi y$; then $\frac{dz}{dy} = \phi' y$, and $\frac{d^2z}{dy^2} = \phi'' y$. Also assume $\frac{dz}{dx}$ an arbitrary function of y, or $\frac{dz}{dx} = \psi y$; then $\frac{d^2z}{dxdy} = \psi y$; or $\frac{d^2z}{dx^2}$ may be expressed in terms of ϕy and of ψy.

By successive differentiation, it may be shown that all the succeeding coefficients of Maclaurin's theorem may be expressed in terms of the same arbitrary functions of y.

Repeating the process, it may be shown that, *in general*, the canonical primitive of a partial equation of the nth order contains n arbitrary functions.

Ex. $\frac{d^2z}{dx^2} = c^2\frac{d^2z}{dy^2}$.

Here z and $\frac{dz}{dx}$ are to be assumed arbitrary functions of y; and by successive differentiation we have

$$\left.\begin{aligned}
\frac{d^3z}{dx^3} &= c^2\frac{d^3z}{dy^2dx} = c^2\frac{d^2\frac{dz}{dx}}{dy^2} \\
\frac{d^4z}{dx^4} &= c^2\frac{d^4z}{dy^2dx^2} = c^2\frac{d^2\frac{d^2z}{dx^2}}{dy^2} = c^4\frac{d^4z}{dy^4} \\
\frac{d^5z}{dx^5} &= c^2\frac{d^5z}{dy^2dx^3} = c^2\frac{d^2\frac{d^3z}{dx^3}}{dy^2} = c^4\frac{d^4\frac{dz}{dx}}{dy^4}
\end{aligned}\right\}\text{; which all depend}$$

. . . .

upon $\frac{dz}{dx}$ and $\frac{d^2z}{dy^2}$ alone, and consequently by Maclaurin's theorem, we have

$$z = \phi y + \psi y \frac{x}{1} + c^2\varphi'' y \frac{x^2}{1.2} + c^2\psi'' y \frac{x^3}{1.2.3} + c^4\phi'''' y \frac{x^4}{1.2.3.4}$$
$$+ c^4\psi'''' y \frac{x^5}{1.2.3.4.5} + \cdots$$

To confirm this solution; the primitive of the example is $z = \varphi(y + cx) + \psi(y - cx)$ (ch. 8. praxis 34.), and developing by means of Maclaurin's theorem

$$\left.\begin{aligned} z = {} & \phi y + c\phi' y \frac{x}{1} + c^2\varphi'' y \frac{x^2}{1.2} + \cdots \\ & + \psi y - c\psi' y \frac{x}{1} + c^2\psi'' y \frac{x^2}{1.2} - \cdots \end{aligned}\right\}; \text{ which agrees with}$$

the former, if φy be substituted for $\varphi y + \psi y$; and ψy, for $c(\phi' y - \psi y)$.

21. Partial equations may be sometimes integrated in series by the method of Indeterminate Coefficients.

Ex. 1. $(x + y)^2 \frac{d^2z}{dxdy} + m(x + y)\frac{dz}{dx} + m(x + y)\frac{dz}{dy} \cdots$
$+ nz = 0.$

Since x and y are similarly involved, assume

$z = \text{A}(x+y)^\alpha\phi x + \text{B}(x+y)^{\alpha+1}\phi' x + \text{C}(x+y)^{\alpha+2}\varphi'' x \cdots$; $\therefore$

$$\left.\begin{array}{llll} n\text{A}(x+y)^\alpha\phi x + & n\text{B}(x+y)^{\alpha+1}\phi' x + & n\text{C}(x+y)^{\alpha+2}\varphi'' x + \cdots & \\ + \quad 2m\alpha\text{A} + & 2m(\alpha+1)\text{B} + & 2m(\alpha+2)\text{C} + \cdots & \\ + & m\text{A} + & m\text{B} + \cdots & \\ + \alpha(\alpha-1)\text{A} + & (\alpha+1)\alpha\text{B} + & (\alpha+2)(\alpha+1)\text{C} + \cdots & \\ + & \alpha\text{A} + & (\alpha+1)\text{B} + \cdots & \end{array}\right\} = 0$$

This is satisfied by A, indeterminate; and

$$n + 2m\alpha + \alpha^2 - \alpha = 0, \text{ or } \alpha = \frac{1-2m}{2} \pm \sqrt{\frac{1}{4} - m + m^2 - n};$$

and the remaining coefficients give

$$\left.\begin{aligned} \text{B} &= -\frac{m+\alpha}{2(m+\alpha)}\text{A}; & \text{D} &= -\frac{m+\alpha+2}{3(2m+2\alpha+2)}\text{C} \\ \text{C} &= -\frac{m+\alpha+1}{2(2m+2\alpha+1)}\text{B}; & \text{E} &= -\frac{m+\alpha+3}{4(2m+2\alpha+3)}\text{D} \end{aligned}\right\}, \text{ from}$$

which the law is sufficiently manifest.

This series will terminate when $m + a + i = 0$, *i. e.* when $\frac{1}{2} + i \pm \sqrt{\frac{1}{4} - m + m^2 - n} = 0$; which can only happen when $\frac{1}{4} - m + m^2 - n$ is a square.

Having found a similar series for z in terms of ψy, the value of z will be under the form

$z = \text{A}(x+y)^{\alpha}(\phi x + \psi y) + \text{B}(x+y)^{\alpha+1}(\phi' x + \psi' y) + \cdots$; which is the canonical primitive, because it contains two arbitrary functions.

Cor. 1. If $a = -m$, *i. e.* if $n = m^2 - m$,

$z = \text{A}(x+y)^{-m}(\phi x + \psi y)$.

Cor. 2. If $a = -m-1$, *i. e.* if $n = (m+1)(m-2)$,

$z = (x+y)^{-m-1}(\phi x + \psi y) - \frac{(x+y)^{-m}}{2}(\phi' x + \psi' y)$. (Ch. 8. 51. cor.)

Give these arbitrary functions any particular form, such as $\phi x = ax^3$ and $\psi y = by^2$; and it will appear by differentiation, that the resulting values of z satisfy the example.

Ex. 2. $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0.$

To develope v, which is a function of (x, y, z), in a series ascending by the powers of z, assume

$v = \text{v} + \text{v}'z + \text{v}''z^2 + \text{v}'''z^3 + \cdots$ where v, v′, v″ ⋯ are functions of (x, y); then

$$\left.\begin{aligned} \frac{d^2v}{dx^2} &= \frac{d^2\text{v}}{dx^2} + \frac{d^2\text{v}'}{dx^2}z + \frac{d^2\text{v}''}{dx^2}z^2 + \cdots \\ \frac{d^2v}{dy^2} &= \frac{d^2\text{v}}{dy^2} + \frac{d^2\text{v}'}{dy^2}z + \frac{d^2\text{v}''}{dy^2}z^2 + \cdots \\ \frac{d^2v}{dz^2} &= 1.2\text{v}'' + 2.3\text{v}'''z + 3.4\text{v}^{\text{iv}}z + \cdots \end{aligned}\right\} \text{; and by substitution,}$$

$$\frac{d^2\text{v}}{dx^2} + \frac{d^2\text{v}}{dy^2} + 1.2\text{v}'' + \left\{\frac{d^2\text{v}'}{dx^2} + \frac{d^2\text{v}'}{dy^2} + 2.3\text{v}'''\right\}z \quad . \quad . \quad . \quad .$$

$$+ \left\{\frac{d^2\text{v}''}{dx^2} + \frac{d^2\text{v}''}{dy^2} + 3.4\text{v}^{\text{iv}}\right\}z^2 + \ldots = 0\text{; whence}$$

$$\text{v}'' = -\frac{1}{1.2}\left\{\frac{d^2\text{v}}{dx^2} + \frac{d^2\text{v}}{dy^2}\right\}$$

$$\text{v}''' = -\frac{1}{2.3}\left\{\frac{d^2\text{v}'}{dx^2} + \frac{d^2\text{v}'}{dy^2}\right\}$$

$$v^{iv} = -\frac{1}{3.4}\left\{\frac{d^2v''}{dx^2} + \frac{d^2v''}{dy^2}\right\} = \frac{1}{1.2.3.4}\left\{\frac{d^4v}{dx^4} + \frac{2d^4v}{dx^2dy^2} + \frac{d^4v}{dy^4}\right\}$$

$\cdots$ which are the required coefficients of z^2, z^3, $\cdots$ in terms of the two arbitrary functions v and v'.

By the same method, as in ex. 1, Euler shows that

$$\left.\begin{aligned}\frac{a^2}{b^2}\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} - \frac{2m}{x}\frac{dz}{dx} = 0\\ \frac{d^2z}{dy^2} - \frac{b^2}{a^2}\frac{d^2z}{dx^2} + \frac{2m}{y}\frac{dz}{dy} = 0\end{aligned}\right\}$$ are integrable when m is

an integer, or $2m$ an even number.

He integrates the form $\frac{d^2z}{dy^2} = x\frac{d^2z}{dx^2}$ by substituting $z = e^{ay}v$, where $v = fx$; then $a^2v = \frac{xd^2v}{dx^2}$. Again, substituting $v = e^{\int pdx}$, the equation is reduced to $\frac{a^2dx}{x} = dp + p^2dx$; which is Riccati's form. These, with other forms which Euler has integrated in tome iii. ch. 10, we shall pass over to notice a peculiarity which Poisson observes belongs to all partial equations in which the highest fluxional coefficient of z with respect to one of its variables is of a higher order than with respect to the other. These cases form an exception to the proposition of art. 20, that the canonical primitive of an equation of the nth order contains n arbitrary functions.

22. *Required to integrate in a series* $\frac{d^2z}{dx^2} = \frac{dz}{dy}$.

To develope z in a series ascending by the powers of x, assume $z = Y + Y'x + Y''x^2 + Y'''x^3 + \cdots$ then we have

$$\left.\begin{aligned}1.2Y'' + 2.3Y'''x + 3.4Y^{iv}x^2 + \cdots\\ -\frac{dY}{dy} - \frac{dY'}{dy}x - \frac{dY''}{dy}x^2 - \cdots\end{aligned}\right\} = 0;$$ wherefore

$$Y'' = \frac{1}{1.2}\frac{dY}{dy};\ Y''' = \frac{1}{2.3}\frac{dY'}{dy};\ Y^{iv} = \frac{1}{3.4}\frac{dY''}{dy};\ \cdots$$ or all the coefficients may be expressed in terms of Y and Y'.

Let $Y = \phi y$ and $Y' = \psi y$; then

$$z = \phi y + x\psi y + \frac{x^2}{1.2}\phi' y + \frac{x^3}{1.2.3}\psi y \quad \cdots \cdots$$

$$+\frac{x^4}{1.2.3.4}\phi''y + \frac{x^5}{1.2.3.4.5}\psi''y + \ldots$$

If z be developed in a series ascending by the powers of y, by assuming
$z = \mathrm{x} + \mathrm{x}'y + \mathrm{x}''y^2 + \ldots$; there will result

$$z = \phi x + \frac{y}{1}\varphi''x + \frac{y^2}{1.2}\phi^{\mathrm{iv}}x + \frac{y^3}{1.2.3}\phi^{\mathrm{vi}}x + \ldots$$

This solution differs materially from the former, inasmuch as it contains only one arbitrary function. The fact is, that the number of the arbitrary constants will always be the same as the order of the fluxional coefficient in terms of whose variable the series is developed.

It is remarkable, that these two solutions may be made to coincide.

For let $\left.\begin{aligned}\varphi y &= \mathrm{A} + \mathrm{B}y + \frac{\mathrm{C}y^2}{1.2} + \frac{\mathrm{D}y^3}{1.2.3} + \ldots\\ \psi y &= \mathrm{A}' + \mathrm{B}'y + \frac{\mathrm{C}'y^2}{1.2} + \frac{\mathrm{D}'y^3}{1.2.3} + \ldots\end{aligned}\right\}$; then from the first, we have

$$z = \mathrm{A} + \mathrm{A}'x + \frac{\mathrm{B}x^2}{1.2} + \frac{\mathrm{B}'x^3}{1.2.3} + \frac{\mathrm{C}x^4}{1.2.3.4} + \frac{\mathrm{C}'x^5}{1.2.3.4.5} + \ldots$$

$$+ y\left\{\mathrm{B} + \mathrm{B}'x + \frac{\mathrm{C}x^2}{1.2} + \frac{\mathrm{C}'x^3}{1.2.3} + \frac{\mathrm{D}x^4}{1.2.3.4} + \frac{\mathrm{D}'x^5}{1.2.3.4.5} + \ldots\right\}$$

$$+ \frac{y^2}{1.2}\left\{\mathrm{C} + \mathrm{C}'x + \frac{\mathrm{D}x^2}{1.2} + \frac{\mathrm{D}'x^3}{1.2.3} + \ldots\right\}$$

$$+ \frac{y}{1.2.3}\left\{\mathrm{D} + \mathrm{D}'x + \frac{\mathrm{E}x^2}{1.2} + \frac{\mathrm{E}'x^3}{1.2.3} + \ldots\right\}$$

$$\ldots$$

$$= \chi x + \frac{y}{1}\chi''x + \frac{y^2}{1.2}\chi^{\mathrm{iv}}x + \frac{y^3}{1.2.3}\chi^{\mathrm{vi}}x + \cdots;$$ which coincides with the second solution.

Generally, if an equation is of the mth order with respect to one of its fluxional coefficients; and of the nth, with respect to the other, where $n < m$, its canonical primitive may be always reduced to a form which shall contain only n arbitrary functions. (Vid. Poisson. Ecole Polytechnique, cah. 13.)

It appears then from this result that the arbitrary functions in the primitives of partial equations are not in all

respects analogous to the arbitrary constants in total equations.

23. PRAXIS.

1. $dy + xdy - mydx = 0 \therefore \ldots$

$$y = b\left\{1 + m\frac{x-a}{1+a} + \frac{m(m-1)}{2}\frac{(x-a)^2}{(1+a)^2}\cdots\right\} = c(1+x)^m.$$

2. $y(dx+dy) = dx \therefore y = 2^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{2x}{1.3} + \frac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3.6} + \frac{2x^2}{5.27} + \cdots$

3. $dy = (x^2+y^2)dx \therefore v = \mathrm{A}t + \frac{\mathrm{B}}{2}t^2 + \frac{\mathrm{C}}{2.3}t^3 + \frac{\mathrm{D}}{2.3.4}t^4 \ldots$

where $\mathrm{A} = a^2 + b^2$; $\mathrm{B} = 2(a + a^2b + b^3)$;
$\mathrm{C} = 2(1 + 2ab + a^4 + 4a^2b^2 + 3b^4)$; ...

4. $dy = (x^n + cy)dx \therefore v = \mathrm{A}t + \mathrm{B}t^2 + \mathrm{C}t^3 + \mathrm{D}t^4 \ldots$ where
$\mathrm{A} = a^n + cb$; $\mathrm{B} = na^{n-1} + ca^n + c^2b$;
$\mathrm{C} = n(n-1)a^{n-2} + nca^{n-1} + c^2a^n + c^3b$;
$\mathrm{D} = n(n-1)(n-2)a^{n-3} + n(n-1)ca^{n-2} \ldots$
$+ nc^2a^{n-1} + c^3a^n + c^4b$; ...

5. $1 + 2mxy - y^2 + nx^2\frac{dy}{dx} = 0 \therefore$

$$y = \cfrac{1}{1-mx}$$

$$\cfrac{}{1 + \frac{m-n}{2}x}$$

$$\cfrac{}{1 - \frac{m+n}{2}x}$$

$$\cfrac{}{1 + \frac{m-2n}{2}x}$$

$$\cfrac{}{1 - \frac{m+2n}{2}x}$$

$$\cfrac{}{1 + \frac{m-3n}{2}x}$$

$$\cfrac{}{1 - \ldots}$$

6. $x^2d^2y - xdxdy + ax^nydx^2 = bx^mdx^2$

$y = \text{A}x^m + \text{B}x^{m+n} + \text{C}x^{m+2n} + \cdots$, where $\text{A} = \dfrac{b}{m(m-2)}$;

$\text{B} = \dfrac{-a\text{A}}{(m+n)(m+n-2)}$; $\text{C} = \dfrac{-a\text{B}}{(m+2n)(m+2n-2)}$; $\cdots$

which is a particular primitive.

7. $d^2y + b^2yx^{-\frac{12}{5}}dx^2 = 0$ ∴

$$y = c\left\{x^{\frac{3}{5}} - \frac{3x}{5^2\beta^2}\right\}\sin.(c' - 5\beta x^{-\frac{1}{5}}) \quad . \quad . \quad . \quad . \quad . \quad .$$

$$- \frac{3cx^{\frac{4}{5}}}{5\beta}\cos.(c' - 5\beta x^{-\frac{1}{5}}).$$

8. $d^2y - b^2yx^{-\frac{12}{7}}dx^2 = 0$ ∴

$$y = \text{A}e^{7bx^{\frac{1}{7}}}\left\{x^{\frac{3}{7}} - \frac{4.3}{2.7b}x^{\frac{2}{7}} + \frac{5.3}{7^2b^2}x^{\frac{1}{7}} - \frac{5.3.1}{7^3b^3}\right\} \quad . \quad . \quad .$$

$$+ \text{A}'e^{-7bx^{\frac{1}{7}}}\left\{x^{\frac{3}{7}} + \frac{4.3}{2.7b}x^{\frac{2}{7}} + \frac{5.3}{7^2b^2}x^{\frac{1}{7}} + \frac{5.3.1}{7^3b^3}\right\}.$$

9. $x^2d^2y - xdxdy + ax^nydx^2 = bx^mdx^2$ ∴

$y = \text{A}x^m + \text{B}x^{m+n} + \text{C}x^{m+2n} + \ldots$; where

$\text{A} = \dfrac{b}{m(m-2)}$; $\text{B} = -\dfrac{a\text{A}}{(m+n)(m+n-2)}$;

$\text{C} = -\dfrac{a\text{B}}{(m+2n)(m+2n-2)}$; $\cdots$

10. $d^2y + \dfrac{aydx^2}{x^3} = 0$ ∴

$$q = \text{A}\left\{1 - \frac{a}{1.2x} + \frac{a^2}{1.2^23x^2} - \frac{a^3}{1.2^2.3^2.4x^3} + \ldots\right\}; \text{ where}$$

$$p = \text{A}'x + \frac{c'}{x} + \frac{\text{D}'}{x^2} + \ldots$$

$\text{A}' = \dfrac{\text{A}}{a}$; $\text{C}' = -\dfrac{3a\text{A}}{1^3.2^2}$; $\text{D}' = \dfrac{14a^2\text{A}}{1^3.2^3.3^2}$; $\text{E}' = -\dfrac{70a^3\text{A}}{1^3.2^3.3^3.4^2}$.

11. $x^2(1+bx^2)d^2y + x(ex^2-5)dxdy + (5+gx^2)ydx^2 = 0$ ∴

$$\left.\begin{aligned} q &= \text{A}x^5 + \text{B}x^7 + \text{C}x^9 + \cdots \\ p &= \text{A}'x + \text{B}'x^3 + \text{C}'x^5 + \cdots \end{aligned}\right\} \therefore \text{ \&c.}$$

12. $d^2y + \frac{2dxdy}{x} - \frac{b^2dx^2}{x^4} = 0 \therefore y = \text{A}\sin.\left(\frac{b}{x} + c\right)$.

13. $x^2(a - bx)d^2y - 2x(2a - bx)dxdy + 2(3a - bx)ydx^2$
$= 6a^2dx^2y = a + bx + \frac{(\text{A} + \text{B}x)x^2}{a - bx}$.

14. $x^2(1 + bx^2)d^2y - x(5 + 7bx^2)dxdy + 5(1 + 3bx^2)ydx^2$
$= 0$, $y = cx(1 + 2bx^2) + c'x^5$; which gives
$y = cx(1 + 2bx^2)$, $y = c'x$ and $y = cx(1 + bx^2)^2$ for particular values.

15. $\frac{d^2r}{d\theta^2} + 2\tan.\theta\frac{dr}{d\theta} + nr = 0$; or

$2\cos.\theta\frac{d^2r}{d\theta^2} + 4\sin.\theta\frac{dr}{d\theta} + 2n\cos.\theta.r = 0.$

Assume $r = \text{A}\cos.\alpha\theta + \text{B}\cos.(\alpha+2)\theta + \text{c}\cos.(\alpha+4)\theta + \ldots$;
and when $n = m^2 - 1$
$r = \text{A}(m\cos m\theta\cos.\theta + \sin.m\theta\sin.\theta)$
$+ \text{A}'(m\sin.m\theta\cos.\theta - \cos.m\theta\sin.\theta)$.

16. $\frac{d^2z}{dxdy} + \frac{m}{x+y}\left\{\frac{dz}{dx} + \frac{dz}{dy}\right\} = 0 \therefore$
$z = \frac{a}{4}(x^4 - y^4) + \frac{b}{3}(x^3 + y^3)$.

17. $\frac{d^2z}{dy^2} = \frac{d^2z}{dx^2} - \frac{2a^2z}{\cos.^2(ax+b)} \therefore$
$z = a\tan.(ax + b)\{\phi(x + y) + \psi(x - y)\}$
$+ \phi'(x + y) + \psi(x - y)$.

END OF PART SECOND.

THE

CALCULUS OF VARIATIONS.

PART THIRD.

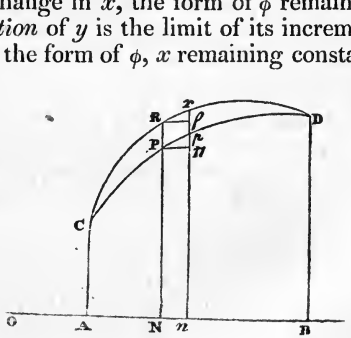
CHAPTER I.

1. In that branch of the fluxional calculus which, for distinction, is called the calculus of variations, the form of the function as well as its principal variable is made to change. It includes a large class of problems, in which the nature of the function is to be determined from the conditions of the question. We shall first establish the general principles of the science, and then apply them to the solution of a few of these problems.

2. *Def.* The *variation* of a function is the limit of its increment arising solely from a change in the form of the function.

Thus; if $y = \phi x$, the *fluxion* of y is the limit of its increment arising from a change in x, the form of ϕ remaining constant; but the *variation* of y is the limit of its increment arising from a change in the form of ϕ, x remaining constant.

3. To give a geometrical illustration of the above definition; let ON, NP be the co-ordinates of a curve CPD, whose equation is $y = \phi x$; and when ON becomes on, suppose CPD to become CRD; draw



the ordinate np and produce NP, np to meet CRD in R and r; also draw Pπ, Rρ perpendicular on nr. Then, if Nn be indefinitely diminished, PR is the variation of NP on the scale that Nn is the fluxion of AN; also the variation of CP is the limit of (CR − CP). If the figure revolve round AB as an axis, the variation of surface CP = the limit of (surface CR − surface CP).

The points C and D, which are either given, or are to be determined from the data of the problem, are in this case called the *limits* of the curve.

4. To distinguish the variation of y from its fluxion, it is denoted by δy.

Thus, to give another illustration of the definition; let (x, y, z) be any point on the surface of a body; then its motion, if the body be at rest, depends upon dx, dy and dz; but if the body move, the motion of the point arising from this circumstance alone depends upon δx, δy and δz. The former depend solely upon the figure of the surface, and are to be determined from its equation; the latter depend upon its motion, and are certain functions of the time.

5. Since the variation is a function of the variables, it may be differentiated; and thus we may have $d\delta y$, $d^2\delta y$, $\cdots$ $d^n\delta y$; as also $d\delta x$, $d^2\delta x$, $\cdots$ $d^n\delta x$.

Also the variation of the fluxion of either variable may be taken. Thus, to find the variation of dy, since $p\pi = dy$ before the change of the curve and $r\rho = dy$ after the change, we have δdy = the limit of $(r\rho - p\pi)$.

Similarly we may have δd^2y, δd^3y, $\cdots$ δd^ny.

For the sake of brevity, we here represent both the fluxion and the variation by the small increments which are to be indefinitely diminished in order to find the limit. This, as has been observed, vol. i. ch. 9. 16, changes the scale of representation, but does not affect the result.

6. Since the variation is a species of fluxion, it is manifest that the algorithm of this calculus must be the same in every respect as that of the calculus of fluxions; so that by the rules established in vol. i. ch. 1, we may find the variation of any simple function, algebraical or transcendental.

Thus, $\delta.y^n = ny^{n-1}\delta y$; $\delta.xy = y\delta x + x\delta y$ and $\delta.a^x = \text{A}a^x\delta x$.

If it be required to prove that $\delta.y^n = ny^{n-1}\delta y$; let $y = \phi x$, and suppose φ to become ϕ'; then $y + \delta y = \phi' x$ in the limit; and $\delta.y^n$, which = the limit of $((y+\delta y)^n - y^n)$, $= ny^{n-1}\delta y$.

Hence, generally, in taking the variation of any function of y, the coefficient of δy is the same as the fluxional coefficient of dy.

Cor. If $u = \text{F}(x, y, p, q, \ldots)$ and we have

$$du = \text{M}dx + \text{N}dy + \text{P}dp + \text{Q}dq + \ldots$$

then $\delta u = \text{M}\delta x + \text{N}\delta y + \text{P}\delta p + \text{Q}\delta q + \ldots$ . . . . . .

$$= \frac{du}{dx}\delta x + \frac{du}{dy}\delta y + \frac{du}{dp}\delta p + \ldots$$

7. *The variation of the fluxion of a function is equal to the fluxion of its variation; or* $\delta dy = d\delta y$.

For, let $y = \phi x$; then ϕ becoming ϕ', $\delta y = \phi' x, - \phi x$, which assume $= \psi y$. Substitute in this equation $y + dy$ for y, and we have $\delta(y + dy)$, or $\delta y + \delta dy = \psi(y + dy)$; wherefore $\delta dy = \psi(y + dy) - \psi y = d\psi y = d\delta y$.

Otherwise: Constructing the figure as in art. 3, the variation of NP is PR; and consequently the fluxion of the variation of NP is the limit of $(pr - \text{PR}) = pr - \pi\rho = r\rho - p\pi =$ the variation of the fluxion of NP.

8. *The variation of the nth fluxion of a function is equal to the nth fluxion of its variation; or* $\delta d^n y = d^n \delta y$.

For $\delta d^2 y = \delta ddy = d\delta dy = dd\delta dy = d^2\delta y$.

Generally, $\delta d^n y = d\delta d^{n-1} y = d^2\delta d^{n-2} y = \cdots d^n \delta y$.

The truth of these propositions follows immediately from the general principle that when the symbols indicate a succession of independent operations, the result is not affected by the order of the operations.

Upon the same principle $\delta \int \text{v} = \int \delta \text{v}$.

Or thus: $d \int \text{v} = \text{v}$; wherefore $\delta \text{v} = \delta d \int \text{v} = d\delta \int \text{v}$, and integrating, $\int \delta \text{v} = \delta \int \text{v}$.

Again, to show that $\delta \iint \text{v} = \iint \delta \text{v}$, we have

$\delta \iint \text{v} = \int \delta \int \text{v} = \iint \delta \text{v}$.

Generally, $\delta \int^n \text{v} = \int^n \delta \text{v}$.

Having established the truth of this elementary proposition, we shall now proceed to deduce formulæ for the variation of such functions as occur in solving the problems which belong to this calculus.

9. If $u = \text{F}(x, y, z, v, \cdots)$ where x is the principal variable, we shall always substitute

$$p = \frac{dy}{dx}, \; q = \frac{d^2 y}{dx^2}, \; r = \frac{d^3 y}{dx^3}, \cdots$$

$$p' = \frac{dz}{dx}, q' = \frac{d^2z}{dx^2}, r' = \frac{d^3z}{dx^3}, \cdots$$

$$p'' = \frac{dv}{dx}, q'' = \frac{d^2v}{dx^2}, r'' = \frac{d^3v}{dx^3}, \cdots$$

10. *Required to find the variation of the fluxional coefficients of* $y = \varphi x$.

In consequence of a change in the form of φ, let y become y', and x become x'; then $\delta p =$ the limit of $\frac{dy'}{dx'} - \frac{dy}{dx}$. But $y' = y + \delta y$ and $x' = x + \delta x$; wherefore

$$\delta p = \frac{d(y+\delta y)}{d(x+\delta x)} - \frac{dy}{dx} =, \text{ in the limit,}$$

$$\frac{dx d\delta y - dy d\delta x}{dx^2} = \frac{d\delta y}{dx} - p\frac{d\delta x}{dx}.$$

Similarly $\delta q = \frac{dp'}{dx'} - \frac{dp}{dx} = \frac{d\delta p}{dx} - q\frac{d\delta x}{dx}$

and $\delta r = \frac{d\delta q}{dx} - r\frac{d\delta x}{dx}; \cdots$

11. In the fluxional calculus, when the higher orders of fluxions are introduced, it is convenient to consider as constant the first fluxion of one of the variables, as x; and this does not render the results less general, since the fluxional formula may be always changed into one in which both dx and dy shall vary (ch. 6. 36). In the same manner, in the calculus of variations we may consider one of the variables, as x, to have no variation; and taking the same variable x to have no variation in the one calculus whose fluxion is constant in the other, we can as before change our hypothesis if it be necessary into one in which both x and y shall vary.

In fig. 3 the curve is supposed to be varied by varying only one of the co-ordinates, as y; and the co-ordinates of R, the corresponding point of the new curve, are $(x, y+\delta y)$; but if we suppose both the co-ordinates to vary, the corresponding point of the new curve is r, and its co-ordinates are $(x + \delta x, y + \delta y)$, where it must be observed that the symbol δy does not represent the same quantity as in the first hypothesis.

12. *Required to find the variation of a function consisting of two variables and of the fluxional coefficients of one of*

them, in terms of the same fluxional coefficients and of the variations of both the variables.

Let the function be $u = \text{F}(x, y, p, q, \cdots)$; and let
$du = \text{M}dx + \text{N}dy + \text{P}dp + \text{Q}dq + \cdots$; then
$\delta u = \text{M}\delta x + \text{N}\delta y + \text{P}\delta p + \text{Q}\delta q + \cdots$ (1).

But $\delta p = \delta\dfrac{dy}{dx} = \dfrac{\delta dy}{dx} - \dfrac{dy\delta dx}{dx^2} = \dfrac{d\delta y}{dx} - p\dfrac{d\delta x}{dx}$, as in art. 10,

$$= \frac{1}{dx}d(\delta y - p\delta x) + q\delta x; \text{ for}$$

$$\frac{1}{dx}d.p\delta x = q\delta x + \frac{pd\delta x}{dx}.$$

Similarly $\delta q = \dfrac{1}{dx}d(\delta p - q\delta x) + r\delta x$

$$= \frac{1}{dx^2}d^2(\delta y - pdx) + rdx$$

$$\delta r = \frac{1}{dx^3}d^3(\delta y - p\delta x) + sdx.$$

$$\cdots = \cdots$$

For the sake of abridgment, let $\delta y - p\delta x = \omega$, or $\delta y = p\delta x + \omega$; then substituting these values in (1), there results

$$\delta u = (\text{M} + \text{N}p + \text{P}q + \text{Q}r + \cdots)\delta x + \text{N}\omega + \frac{\text{P}d\omega}{dx} + \frac{\text{Q}d^2\omega}{dx^2} + \cdots$$

$$= \frac{du\delta x}{dx} + \text{N}\omega + \frac{\text{P}d\omega}{dx} + \frac{\text{Q}d^2\omega}{dx^2} + \cdots$$

Cor. Since $\omega = \delta y - p\delta x$, the variation of u may be divided into two parts; the one depending solely upon the variation of y; the other, upon the variation of x.

13. *Required to find the variation of a function consisting of three variables and of the fluxional coefficients of two of these considered as functions of the third.*

Let $du = \left.\begin{matrix}\text{M}dx + \text{N}dy + \text{P}dp + \text{Q}dq + \cdots \\ + \text{N}'dz + \text{P}'dp' + \text{Q}'dq' + \cdots\end{matrix}\right\}$; then,

proceeding as before, and substituting for the sake of abridgment $\omega' = \delta z - p'\delta x$, there will result

$$\delta u = \frac{du\delta x}{dx} + \mathrm{N}\omega + \frac{\mathrm{P}d\omega}{dx} + \frac{\mathrm{Q}d^2\omega}{dx^2} + \ldots \Big\} + \mathrm{N}'\omega' + \frac{\mathrm{P}'d\omega'}{dx} + \frac{\mathrm{Q}'d^2\omega'}{dx^2} + \ldots \Big\}.$$

This formula may be readily extended to the case of a function consisting of any number of variables and of their fluxional coefficients considered as functions of one of them.

14. *Required to find the variation of a function of three variables, one of which is a function of the other two; the proposed function containing the variables and their fluxional coefficients.*

Let the function be $u = \mathrm{F}(x, y, z, p, q, r, s, t \ldots)$, where $z = f(x, y)$ and $p = \frac{dz}{dx}$, $q = \frac{dz}{dy}$, $r = \frac{d^2z}{dx^2}$, $s = \frac{d^2z}{dxdy}$, $t = \frac{d^2z}{dy^2} \cdots$; then we have

$$\delta u = \frac{du}{dx}\delta x + \frac{du}{dy}\delta y + \frac{du}{dz}\delta z + \frac{du}{dp}\delta p + \cdots$$

But $\delta p = \delta\frac{dz}{dx} = \frac{d\delta z}{dx} - p\frac{d\delta x}{dx}$

$$= \frac{d(\delta z - p\delta x)}{dx} + \frac{dp}{dx}\delta x$$

$$= \frac{d(\delta z - p\delta x - q\delta y)}{dx} + \frac{dp}{dx}\delta x + \frac{dq}{dx}\delta y;$$ for

$\frac{d.q\delta y}{dx} = \frac{dq}{dx}\delta y$ when δy is constant (art. 11).

Substitute $\omega = \delta z - p\delta x - q\delta y$; then

$$\delta p = \frac{d\omega}{dx} + \frac{dp}{dx}\delta x + \frac{dq}{dx}\delta y = \frac{d\omega}{dx} + \frac{dp}{dx}\delta x + \frac{dp}{dy}\delta y.$$

Similarly $\delta q = \frac{d\omega}{dx} + \frac{dq}{dx}\delta x + \frac{dq}{dy}\delta y$.

Also, $\delta r = \delta\frac{dp}{dx} = \frac{d\delta p}{dx} - \frac{dpd\delta x}{dx^2} = \frac{d\delta p}{dx}$

$$= \frac{d^2\omega}{dx^2} + \frac{dr}{dx}\delta x + \frac{dr}{dy}\delta y$$

$$\delta s = \frac{d^2\omega}{dxdy} + \frac{ds}{dx}\delta x + \frac{ds}{dy}\delta y$$
$$\delta t = \frac{d^2\omega}{dy^2} + \frac{dt}{dx}\delta x + \frac{dt}{dy}\delta y$$
$$\cdots = \cdots$$

Wherefore there results by substitution, since

$$\delta z = \omega + \frac{dz}{dx}\delta x + \frac{dz}{dy}\delta y,$$

$$\begin{aligned}\delta u = &\left\{\frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} + \frac{du}{dp}\frac{dp}{dx} + \frac{du}{dq}\frac{dq}{dx}\cdots\right\}\delta x\\ &+ \left\{\frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} + \frac{du}{dp}\frac{dp}{dy} + \frac{du}{dq}\frac{dq}{dy}\cdots\right\}\delta y\\ &+ \frac{du}{dz}\omega + \frac{du}{dp}\frac{d\omega}{dx} + \frac{du}{dq}\frac{d\omega}{dy}\\ &+ \frac{du}{dr}\frac{d^2\omega}{dx^2} + \frac{du}{ds}\frac{d^2\omega}{dxdy} + \frac{du}{dt}\frac{d^2\omega}{dy^2}\\ &+ \cdots\\ &= \frac{\ddot{du}}{dx}\delta x + \frac{\ddot{du}}{dy}\delta y + \frac{du}{dz}\omega\\ &+ \frac{du}{dp}\frac{d\omega}{dx} + \frac{du}{dq}\frac{d\omega}{dy}\\ &+ \frac{du}{dr}\frac{d^2\omega}{dx^2} + \frac{du}{ds}\frac{d^2\omega}{dxdy} + \frac{du}{dt}\frac{d^2\omega}{dy^2}\\ &+ \cdots\end{aligned}$$

It is manifest that this formula also may be extended to a function consisting of any number of variables.

15. In those problems which belong to the calculus of variations, u appears under the form of a fluent as $\int v dx$, where v is a certain function of the variables and of their fluxional coefficients; and this fluent is to be integrated between limits, in fig. 3 between $x = \text{OA}$, $x = \text{OB}$. We shall enclose this in brackets when it is intended to denote that it is to be integrated between the limits.

The problems of this calculus, in which only two variables are concerned, appear under the following form: 'Of all the possible relations between x and y, to find that which gives to $(\int v dx)$ a maximum or minimum value, v being a function of $(x, y, p, q, \cdots)$.' The difficulties with which

we have to contend in solving these problems arise from two distinct causes; the one, the nature of the function v; the other, the nature of the limits between which the fluent is to be integrated. Thus, to take the simplest case for the purpose of illustration; if it be required to draw the shortest line between two given points, the limits are fixed and known; but if it be required to find the shortest line which can be drawn joining the circumferences of two given circles, the position of the limits forms an essential and a distinct part of the problem.

16. It is also manifest that in the case of a maximum or a minimum the whole variation of $\int \mathrm{v}dx$ or $\delta(\int \mathrm{v}dx)$ is equal to nothing upon the same principle that the fluxion of a maximum or a minimum $= 0$. The rules also for distinguishing a maximum from a minimum are the same as those established in the first volume. It is, however, in general unnecessary to apply these rules, as the problems are of such a nature that it can be seen at once whether the result is a maximum or a minimum.

17. *Required to find the variation of* $\int \mathrm{v}dx$, *where* $\mathrm{v} = \mathrm{F}(x, y, p, q, \ldots)$.

$$\delta\int \mathrm{v}dx = \int \delta.\mathrm{v}dx = \int dx\delta\mathrm{v} + \int \mathrm{v}\delta dx = \int dx\delta\mathrm{v} + \mathrm{v}\delta x - \int d\mathrm{v}\delta x$$
$$= \mathrm{v}\delta x + \int(\delta\mathrm{v}dx - d\mathrm{v}\delta x) \quad (1).$$

Let $d\mathrm{v} = \mathrm{M}dx + \mathrm{N}dy + \mathrm{P}dp + \mathrm{Q}dq + \cdots$; then it may be shown as in art. 12, that

$$\delta\mathrm{v} = \frac{\dot{d}\mathrm{v}x\delta}{\dot{d}x} + \mathrm{N}\omega + \frac{\mathrm{P}d\omega}{dx} + \frac{\mathrm{Q}d^2\omega}{dx^2} + \cdots; \text{ where}$$

$\omega = \delta y - p\delta x$; whence $\int(\delta\mathrm{v}dx - d\mathrm{v}\delta x) \cdot \cdot \cdot \cdot \cdot \cdot$

$$= \int dx \left\{ \mathrm{N}\omega + \frac{\mathrm{P}d\omega}{dx} + \frac{\mathrm{Q}d^2\omega}{dx^2} + \cdots \right\}; \text{ and by substitution in}$$

$$(1)\ \delta\int \mathrm{v}dx = \mathrm{v}\delta x + \int dx \left\{ \mathrm{N}\omega + \frac{\mathrm{P}d\omega}{dx} + \frac{\mathrm{Q}d^2\omega}{dx^2} + \cdots \right\} \quad (2).$$

But $\int \mathrm{P}d\omega = \mathrm{P}\omega - \int \omega d\mathrm{P}$

$$\int \mathrm{Q}d^2\omega = \mathrm{Q}d\omega - \int d\mathrm{Q}d\omega = \mathrm{Q}d\omega - \omega d\mathrm{Q} + \int \omega d^2\mathrm{Q}$$
$$\int \mathrm{R}d^3\omega = \mathrm{R}d^2\omega - \int d\mathrm{R}d^2\omega = \mathrm{R}d^2\omega - d\mathrm{R}d\omega + \int d\omega d^2\mathrm{R}$$
$$= \mathrm{R}d^2\omega - d\mathrm{R}d\omega + \omega d^2\mathrm{R} - \int \omega d^3\mathrm{R}$$
$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

Generally $\int \mathrm{T}d^n\omega = \mathrm{T}d^{n-1}\omega - d\mathrm{T}d^{n-2}\omega + \cdots \pm \int \omega d^n\mathrm{T}$,

where $\mathrm{T} = \frac{d\mathrm{v}}{dt}$ and t is the nth fluxional coefficient.

Substituting these values in (2), and collecting quantities involving like symbols, there will result,

$$\left.\begin{aligned}\delta\int \mathrm{v}dx = \mathrm{v}\delta x &+ \left\{\mathrm{P} - \frac{\dot{d}\mathrm{Q}}{dx} + \frac{\dot{d}^2\mathrm{R}}{dx^2} - \frac{\dot{d}^3\mathrm{S}}{dx^3} + \dots\right\}\omega \\ &+ \left\{\mathrm{Q} - \frac{\dot{d}\mathrm{R}}{dx} + \frac{\dot{d}^2\mathrm{S}}{dx^2} - \dots\right\}d\omega \\ &+ \left\{\mathrm{R} - \frac{\dot{d}\mathrm{S}}{dx} + \dots\right\}d^2\omega \\ &+ \left\{\mathrm{S} - \dots\right\}d^3\omega \\ &+ \int\left\{\mathrm{N} - \frac{\dot{d}\mathrm{P}}{dx} + \frac{\dot{d}^2\mathrm{Q}}{dx^2} - \frac{\dot{d}^3\mathrm{R}}{dx^3} + \dots\right\}\omega dx + \mathrm{K}\end{aligned}\right\}(\mathrm{A})$$

where K is the result of all the corrections.

Cor. 1. $\int(\delta\mathrm{v}dx - d\mathrm{v}\delta x) = \mathrm{L} + \int\left\{\mathrm{N} - \frac{\dot{d}\mathrm{P}}{dx} + \dots\right\}\omega dx,$

where L expresses the sum of all the terms of (A) except the last, $\mathrm{v}\delta x$ being transposed to the other side of the equation.

Cor. 2. If we suppose $\omega = 0$, there results $\delta\int \mathrm{v}dx = \mathrm{v}\delta x$; which ought to be the result, because in this case $\delta y - p\delta x = 0$, or we establish the same relation between δy and δx as exists between dy and dx; and consequently $\delta\int \mathrm{v}dx$ is the same as $d\int \mathrm{v}dx$ or $\mathrm{v}dx$.

Cor. 3. If we replace ω by its value $\delta y - p\delta x$, it will appear that the whole variation consists of two parts: the one wholly due to the variation of x; the other, to the variation of y.

Cor. 4. If we suppose x to have no variation, $\omega = \delta y$; and the whole variation is found by substituting δy, $d\delta y$, ... for ω, $d\omega$, ...

Cor. 5. If v contain *three* variables x, y, z and their fluxional coefficients p, q, ... p', q', ...; then, substituting $\mathrm{N}' = \frac{d\mathrm{v}}{dz}$, $\mathrm{P}' = \frac{d\mathrm{v}}{dp'}$, $\mathrm{Q}' = \frac{d\mathrm{v}}{dq'}$, ... and also $\omega' = \delta z - p'\delta x$, we may find the variation by adding to the expression already obtained the part which follows $\mathrm{v}\delta x$; N, P, Q, ... and ω being accented.

Cor. 6. If v contain any number of variables, and be

under the form $\mathrm{v} = \mathrm{F}(x, y, z, v, \ldots p, q \ldots, p', q', \ldots p'', q'', \ldots)$, the variation may be found by similar additions.

It is observable that M, which is the fluxional coefficient of v when differentiated with respect to x, does not enter into (A): the cause of this will be seen at art. 25.

18. It is worthy of remark, that in the formula (A) the coefficients of the parts under the sign of integration, when made $= 0$, are the very equations of condition deduced Part 2, chap. 7, 13, that the function $\mathrm{v}dx$ may be an exact fluxion.

These equations may be deduced immediately from the theory of variations.

For, let $du = \mathrm{v}dx$; then $\delta \int \mathrm{v}dx = \delta \int du = d\delta \iint \mathrm{v}dx$, an exact fluxion; which shows, that when we have brought from under the sign of integration all the integrable quantities, the sum of the remainder must $= 0$; or

$$\left\{ \mathrm{N} - \frac{d\mathrm{P}}{dx} + \ldots \right\} \omega + \left\{ \mathrm{N}' - \frac{d\mathrm{P}'}{dx} + \ldots \right\} \omega' + \ldots = 0.$$ But $\omega, \omega', \ldots$ contain the indeterminate quantities $\delta x, \delta y, \delta z, \ldots$; wherefore $\mathrm{N} - \frac{d\mathrm{P}}{dx} + \ldots = 0$; $\mathrm{N}' - \frac{d\mathrm{P}'}{dx} + \ldots = 0$; ...

The other equations of condition that the function may be an exact fluxion of the nth order may be likewise deduced by means of the theory of variations; but we shall not enter into the calculation.

19. *Required to find* $\delta(\int \mathrm{v}dx)$ *when* v *is a function of two variables and of their fluxional coefficients.*

Let $\mathrm{v}_{\prime}$, $y_{\prime}$, $\mathrm{P}_{\prime}$, ... represent the values of v, y, P, ... at the first limit;

$\mathrm{v}_{\prime\prime}$, $y_{\prime\prime}$, $\mathrm{P}_{\prime\prime}$, ... their values at the second limit.

Then

$$\delta(\int \mathrm{v}dx) = \mathrm{v}_{\prime\prime}\delta x_{\prime\prime} - \mathrm{v}_{\prime}\delta x_{\prime} + \left\{ \mathrm{P}_{\prime\prime} - \frac{d\mathrm{Q}_{\prime\prime}}{dx} + \frac{d^2\mathrm{R}_{\prime\prime}}{dx^2} - \ldots \right\} \omega_{\prime\prime} - \left\{ \mathrm{P}_{\prime} - \frac{d\mathrm{Q}_{\prime}}{dx} + \frac{d^2\mathrm{R}_{\prime}}{dx^2} - \ldots \right\} \omega_{\prime} \Bigg\} (1).$$

$$+ \ldots + \left(\int \left\{ \mathrm{N} - \frac{d\mathrm{P}}{dx} + \frac{d^2\mathrm{Q}}{dx^2} - \ldots \right\} \omega dx \right) (2).$$

This consists of two parts, which are wholly different:

the first depends upon the variation of x and y *at the limits*; the second is included under the symbol of integration: it represents the fluent of the variation, or the variation of the fluent *between the limits*; and it is enclosed in brackets to denote the whole variation between the assigned limits of the function. It is manifest that the former or the *definite* part is wholly independent of the latter or the indefinite part; which, as it includes ω, *i. e.* the *general* values of δx and δy, may be supposed to vary, while the other remains fixed and determinate.

Cor. If the two limits are given as in fig. 3, where they are $x = a$, $x = b$, we have $\delta x_{\prime}$, $dy_{\prime}$, and $\delta x_{\prime\prime}$, $\delta y_{\prime\prime}$, each $= 0$, and consequently

$$\delta \int_a^b \mathrm{V} dx = \int_a^b \left\{ \mathrm{N} - \frac{d\mathrm{P}}{dx} + \frac{d^2\mathrm{Q}}{dx^2} - \ldots \right\} \omega dx.$$

20. In the case of a maximum or a minimum, we have the whole expression $= 0$, and consequently each part $= 0$.

The part (2) cannot $= 0$ unless $\mathrm{N} - \frac{d\mathrm{P}}{dx} + \frac{d^2\mathrm{Q}}{dx^2} - \ldots = 0$;

because ω contains the indeterminate quantities δx and δy. This is Euler's formula *.

The part (1) gives an equation which contains four indeterminate quantities $\delta x_{\prime}$, $\delta y_{\prime}$ and $\delta x_{\prime\prime}$, $\delta y_{\prime\prime}$. If we can eliminate two of these, $\delta y_{\prime}$ and $\delta y_{\prime\prime}$, by means of the conditions of the question, there will result an equation which will divide into two by making the coefficients of $\delta x_{\prime}$, $\delta x_{\prime\prime}$ each $= 0$. These we shall call *equations of limits*.

Cor. 1. If $\mathrm{V} = \mathrm{F}(x, y, z, p, q, \ldots p', q', \ldots)$, the indefinite part becomes

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* Euler obtained it by considering only the indefinite part; but it was Lagrange who gave it its present general form by adding the definite part; and to the honour of Euler it should be recorded, that he was the first to acknowledge the great improvement which his formula received from this addition. "Neque tamen hæ partes absolutæ frustra sunt inventæ, sed singularem præbent usum, ad quem methodus mea prior, quæ tantum æquationem $\mathrm{N} - \frac{d\mathrm{P}}{dx} + \ldots = 0$ suppeditavit, minus est accommodata; quam ob causam hæc methodus illi longe est anteferenda." (Vid. Woodhouse, Isop. Prob. p. 90.)

$$\left\{ \text{N} - \frac{d\text{P}}{dx} + \dots \right\} \omega + \left\{ \text{N}' - \frac{d\text{P}'}{dx} + \dots \right\} \omega' = 0.$$

Since $\omega = \delta y - p\delta x$ and $\omega' = \delta z - p'\delta x$, this is under the form $\text{A}\delta x + \text{B}\delta y + \text{C}\delta z = 0$; which, when y and z are independent of each other, divides into $\text{A}\delta x + \text{B}\delta y = 0$ and $\text{A}\delta x + \text{C}\delta z = 0$.

Cor. 2. If y and z are not independent, but are connected by an equation of condition $\text{F}(x, y, z) = 0$, we may by means of it eliminate both δy and δz from $\text{A}\delta x + \text{B}\delta y + \text{C}\delta z = 0$, and there will result, as before, two equations which express the relation between δx, δy and δz.

Similar equations may be deduced when $\text{V} = \text{F}(x, y, z, v \dots p, q \dots p', q' \dots p'', q'' \dots)$.

21. In the solution of problems, if the limits are two curves, we shall represent the co-ordinates of the curve at the first limit by x', y'; and at the second by x'', y'': these $= x_I$, y_I and x_{II}, y_{II} respectively: but it does not follow that $\frac{dy'}{dx'}$ and $\frac{dy_I}{dx_I}$ are equal; for the one marks the position of the tangent of the limiting curve; and the other, the position of the tangent of the required curve.

We shall denote $\frac{dy'}{dx'}$, $\frac{dy''}{dx''}$, which belong to the limiting curves, by p_1, p_2; and $\frac{dy_I}{dx_I}$, $\frac{dy_{II}}{dx_{II}}$, which belong to the required curve, by p_I, p_{II} ... as before.

22. Required to find the equations of limits of $\delta(\int v dx) = 0$.

The whole equation of limits is

$$\left.\begin{aligned} &\text{V}_{II}\delta x_{II} - \text{V}_I \delta x_I \\ &+ \left\{ \text{P}_{II} - \frac{d\text{Q}_{II}}{dx} + \dots \right\} \omega_{II} \\ &- \left\{ \text{P}_I - \frac{d\text{Q}_I}{dx} + \dots \right\} \omega_I \\ &+ \dots \end{aligned}\right\} = 0.$$

Represent this by $\delta \text{U}_{II} - \delta \text{U}_I = 0$. Suppose the limits to be two curves; of which the first is $\text{L}' = 0$; and the second, $\text{L}'' = 0$. Let x', y' be the co-ordinates of $\text{L}' = 0$ at the first limit; and x'', y'' those of $\text{L}'' = 0$ at the second limit.

Then, since $\text{L}' = \text{F}(x', y')$ and $\text{L}'' = f(x'', y'')$, we have, differentiating

$$\left.\begin{aligned}\frac{d\mathrm{L}'}{dx}+\frac{d\mathrm{L}'}{dy}p_1=0\\ \frac{d\mathrm{L}''}{dx}+\frac{d\mathrm{L}''}{dy}p_2=0\end{aligned}\right\};\ \text{where } p_1=\frac{dy'}{dx'} \text{ and } p_2=\frac{dy''}{dx''}.$$

Also, since $\mathrm{L}' = \mathrm{F}(x_I, y_I)$ and $\mathrm{L}'' = f(x_{II}, y_{II})$, we have, taking the variation,

$$\left.\begin{aligned}\frac{d\mathrm{L}'}{dx}+\frac{d\mathrm{L}'}{dy}\frac{\delta y_I}{\delta x_I}=0\\ \frac{d\mathrm{L}''}{dx}+\frac{d\mathrm{L}''}{dy}\frac{\delta y_{II}}{\delta x_{II}}=0\end{aligned}\right\};\ \text{which equations belong to the re-}$$

quired curve, for it is the only curve which *varies*.

Hence, by elimination, $\left.\begin{aligned}\delta y_I - p_1\delta x_I = 0\\ \delta y_{II} - p_2\delta x_{II} = 0\end{aligned}\right\}$; and combining these with $\delta \mathrm{U}_{II} - \delta \mathrm{U}_I = 0$, we can find either δx_I, δy_I, or δx_{II}, δy_{II}; which will give the conditions that must be fulfilled at the two limits.

If the limits are independent of each other—two curves, for instance, not connected by any equation—we have δx_I, δy_I independent of δx_{II}, δy_{II}, and consequently ω_I independent of ω_{II}; wherefore the equation divides into $\delta \mathrm{U}_I = 0$ and $\delta \mathrm{U}_{II} = 0$; or $\mathrm{V}_I\delta x_I + \mathrm{P}_I\omega_I - \cdots = 0$ and $\mathrm{V}_{II}\delta x_{II} + \mathrm{P}_{II}\omega_{II} - \cdots = 0$.

Euler has deduced from the general equation

$$\mathrm{N} - \frac{d\mathrm{P}}{dx} + \cdots = 0$$ certain formulæ, which in particular cases render its integration more easy: these we shall give in the next article.

23. *Formulæ of solution.*

(1). In $d\mathrm{V} = \mathrm{M}dx + \mathrm{N}dy + \mathrm{P}dp + \mathrm{Q}dq \ldots$, suppose all the coefficients $= 0$ except N and P.

Here $d\mathrm{V} = \mathrm{N}dy + \mathrm{P}dp$. Also the equation of the curve is $\mathrm{N} - \frac{d\mathrm{P}}{dx} = 0$; or, since $dx = \frac{dy}{p}$, $\mathrm{N}dy = pd\mathrm{P}$; wherefore by substitution $d\mathrm{V} = pd\mathrm{P} + \mathrm{P}dp = d.\mathrm{P}p \therefore \mathrm{V} = \mathrm{P}p + c$.

Cor. If M is not $= 0$, $\mathrm{V} = \int \mathrm{M}dx + \mathrm{P}p + c$.

(2). Suppose all the coefficients $= 0$ except P and Q.

Here $dv = \mathrm{P}dp + \mathrm{Q}dq$. Also the equation of the curve is $-\frac{d\mathrm{P}}{dx} + \frac{d^2\mathrm{Q}}{dx^2} = 0$; or, integrating, $\mathrm{P} = \frac{d\mathrm{Q}}{dx} + c$; and mul-

tiplying by dp, $\text{P}dp = \frac{dp}{dx}d\text{Q} + cdp = qd\text{Q} + cdp$; and by substitution, $d\text{V} = \text{Q}dq + qd\text{Q} + cdp \therefore \text{V} = \text{Q}q + cp + c'$.

Cor. If M is not $= 0$, $\text{V} = \int \text{M}dx + \text{Q}q + cp + c'$.

(3). Suppose all the coefficients $= 0$ except N, P and Q.

Here $d\text{V} = \text{N}dy + \text{P}dp + \text{Q}dq$. Also the curve's equation is $\text{N} - \frac{d\text{P}}{dx} + \frac{d^2\text{Q}}{dx^2} = 0$; or multiplying by dy, which $= pdx$, $\text{N}dy - pd\text{P} + \frac{pd^2\text{Q}}{dx} = 0$.

But $pd^2\text{Q} = d.pd\text{Q} - dpd\text{Q} = d.pd\text{Q} - qd\text{Q}dx$, or $\text{N}dy = pd\text{P} - \frac{d.pd\text{Q}}{dx} + qd\text{Q}$; and by substitution,

$$d\text{V} = \text{P}dp + pd\text{P} - \frac{d.pd\text{Q}}{dx} + \text{Q}dq + qd\text{Q} \therefore$$

$$\text{V} = \text{P}p + \text{Q}q - \frac{pd\text{Q}}{dx} + c.$$

(4). Suppose all the coefficients $= 0$ except P and R.

Here $d\text{V} = \text{P}dp + \text{R}dr$. Also the curve's equation is $\frac{d\text{P}}{dx} + \frac{d^3\text{R}}{dx^3} = 0$; wherefore $\text{P} = -\frac{d^2\text{R}}{dx^2} + c$, or

$$\text{P}dp = cdp - \frac{qd^2\text{R}}{dx}. \quad \text{But } qd^2\text{R} = d.qd\text{R} - dqd\text{R} \therefore$$

$$\text{P}dp = cdp - \frac{d.qd\text{R}}{dx} + rd\text{R}; \text{ and by substitution we have}$$

$$d\text{V} = cdp + \text{R}dr + rd\text{R} - \frac{d.qd\text{R}}{dx} \text{ and}$$

$$\text{V} = c' + cp + \text{R}r - \frac{qd\text{R}}{dx}.$$

24. We have hitherto supposed the required fluent to be expressed by a single equation; but in problems which relate to lines in space, it consists of two simultaneous and independent equations. (Vol. i. ch. 7. 54).

Let $d\text{V} = \left.\begin{matrix} \text{M}dx + \text{N}dy + \text{P}dp \cdots \\ + \text{N}'dz + \text{P}'dp' \cdots \end{matrix}\right\}$.

To find the two equations in space between x, y and x, z between the limits; for the first, we may suppose z con-

stant; or the required fluxional equation is

$\text{N} - \frac{d\text{P}}{dx} + \cdots = 0$. Similarly, to find the second, y is constant, and we have $\text{N}' - \frac{d\text{P}'}{dx} + \cdots = 0$.

The equations at the limits are to be obtained from the definite part of $\delta(\int \text{V}dx)$, which separates into two as in art. 22.

Cor. Euler has given the following formula for finding the function between the limits in the case in which all the coefficients vanish except M, N and P. (Mémoires de l'Acad. de Petersbourg, tom. 4, p. 23).

Here $\text{N}dx - d\text{P} = 0$ and $\text{N}'dx - d\text{P}' = 0$; but $\frac{dy}{p} = dx = \frac{dz}{p'}$; whence we have $\text{N}dy - pd\text{P} = 0$ and $\text{N}'dz - p'd\text{P}' = 0$; wherefore, by substitution,

$d\text{V} = \text{M}dx + pd\text{P} + \text{P}dp + p'd\text{P}' + \text{P}'dp' = \text{M}dx + d(\text{P}p + \text{P}'p')$.

Substitute $\text{S} = \text{V} - (\text{P}p + \text{P}'p')$; then we have three equations $\text{N}dx = d\text{P}$, $\text{N}'dx = d\text{P}'$ and $\text{M}dx = d\text{S}$; by means of which x, y and z may be each expressed in terms of the same quantity p; and eliminating p, the required equations in space may be found *.

25. *The indefinite part of the whole variation is not affected by the variation of the principal variable.*

For, let $\text{V} = \text{F}(x, y, z, p, q, \ldots p', q', \ldots)$; and substitute

$$\left.\begin{aligned} \text{A} &= \text{N} - \frac{d\text{P}}{dx} + \frac{d^2\text{Q}}{dx^2} - \cdots \\ \text{A}' &= \text{N}' - \frac{d\text{P}'}{dx} + \frac{d^2\text{Q}'}{dx^2} - \cdots \end{aligned}\right\};$$

then, as in art. 17, the indefinite part depends upon the equation $\text{A}\omega + \text{A}'\omega' = 0$ (1).

First, suppose $\delta x = 0$; then $\omega = \delta y$ and $\omega' = \delta z$, and the equation becomes $\text{A}\delta y + \text{A}'\delta z = 0$ (2).

Let the equation between the variables x, y, z be $v = 0$;

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* I recommend the student, before he proceeds to the more complicated forms of the function V, to apply the above formulæ to a few examples: these are placed at the end of the chapter, that the subject may not be interrupted.

then (ex hyp.) $\frac{dv}{dy}\delta y + \frac{dv}{dz}\delta z = 0$; and combining this with (2),

there results $\mathrm{A}\frac{dv}{dz} - \mathrm{A}'\frac{dv}{dy} = 0$ (3).

Next, suppose x to vary; then,

$$\left.\begin{aligned}\frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y + \frac{dv}{dz}\delta z &= 0\\ \frac{dv}{dx} + p\frac{dv}{dy} + p'\frac{dv}{dz} &= 0\end{aligned}\right\}\text{; and consequently}$$

$$\frac{dv}{dy}(\delta y - p\delta x) + \frac{dv}{dz}(\delta z - p'\delta x) = 0, \text{ or } \frac{dv}{dy}\omega + \frac{dv}{dz}\omega' = 0;$$

which combined with (1) gives $\mathrm{A}\frac{dv}{dz} - \mathrm{A}'\frac{dv}{dy} = 0$, which is the same result as before; which shows that the variation of x affects only the equations of limits.

Hence, in calculating the indefinite part, we are at liberty to suppose $\delta x = 0$; in which case $\omega = \delta y$ and $\omega' = \delta z$; and the equation becomes $\mathrm{A}\delta y + \mathrm{A}'\delta z = 0$.

The same demonstration may be extended to the case in which $\mathrm{v} = \mathrm{F}(x, y, z, v \cdots p, q \cdots p', q' \cdots p'', q'' \cdots)$.

Cor. If y and z are independent of each other, we have $\mathrm{A} = 0$ and $\mathrm{A}' = 0$.

26. *The indefinite part of the whole variation is not affected by the variation of the parameters.*

Suppose the function a curve, and let
$\mathrm{v} = \mathrm{F}(x, y, p, q \cdots c, c' \cdots)$, where c, $c' \cdots$ are the parameters; then, these varying, let
$dv = \mathrm{M}dx + \mathrm{N}dy + \mathrm{P}dp + \cdots + \mathrm{C}dc + \mathrm{C}'dc' + \cdots$

Suppose $\delta x = 0$; then it may be shown, as in art. 19, that $\delta(\mathrm{v}dx) = \mathrm{L} + \int dx(\mathrm{C}\delta c + \mathrm{C}'\delta c') \cdots$

$$+ \left(\int \left\{\mathrm{N} - \frac{d\mathrm{P}}{dx} + \cdots\right\}\delta y dx\right).$$

Hence it appears that the required curve depends upon the same part as before, and that the two new terms introduced by the variation of the parameters do not affect the nature of the curve, but together with the part L determine the nature and position of the limits.

27. *Required to find the variation of* $\int \mathrm{v}dx$ *when* $d\mathrm{v} = \mathrm{T}dt$;

t being given by $t = \int z dx$; *and* v *being, as before, a function of* $(x, y, p, q \cdots)$.

It may be shown, as in art. 17, that
$\delta \int v dx = v\delta x + \int (\delta v dx - dv\delta x)$. But $dv = \mathrm{T} dt$; wherefore $\delta v = \mathrm{T}\delta t$, and consequently
$\delta v dx - dv\delta x = \mathrm{T}\delta t dx - \mathrm{T} dt\delta x = \mathrm{T}\delta t dx - \mathrm{T} z dx \delta x$.

To eliminate δt, we have
$\delta t = \delta \int z dx = z\delta x + \int (\delta z dx - dz\delta x)$; whence
$\mathrm{T}\delta t dx - \mathrm{T} z dx \delta x = \mathrm{T} dx \int (\delta z dx - dz\delta x)$; and finally
$\delta \int v dx = v\delta x + \int \mathrm{T} dx \int (\delta z dx - dz\delta x)$.

For the sake of abridgment, substitute $dh = \mathrm{T} dx$; then, integrating by parts,
$\int dh \int (\delta z dx - dz\delta x) = h \int (\delta z dx - dz\delta x) - \int h (\delta z dx - dz\delta x)$.

Also, if $dz = \mathrm{M} dx + \mathrm{N} dy + \mathrm{P} dp + \cdots$; then (art. 17, cor. 1),

$$h \int (\delta z dx - dz\delta x) = h\mathrm{L} + h \int \left\{ \mathrm{N} - \frac{d\mathrm{P}}{dx} + \cdots \right\} \omega dx \ (1).$$

Again, $\int h (\delta z dx - dz\delta x) = h\mathrm{L}' \cdots$

$$+ \int \left\{ h\mathrm{N} - \frac{d.h\mathrm{P}}{dx} + \frac{d^2.h\mathrm{Q}}{dx^2} - \cdots \right\} \omega dx \ (2).$$

Subtracting (2) from (1), and adding $v\delta x$, there will result the required variation.

27. *Required to find* $\delta (\int v dx)$ *when* v *contains, besides the variables and their fluxional coefficients, a fluent of the form* $t = \int z dx$.

Suppose that $\int v dx$ is to be integrated between $x = 0$, $x = a$; which will not affect the indefinite part; and let $\int_0^a z dx = \mathrm{H}$, a constant quantity.

Adopt the same notation as in the last article; also, suppose that $dv = \mu dx + \nu dy + \pi dp + \cdots + \mathrm{T} dt$.

Then the variation

$$= v\delta x + \mathrm{S} + \int \left\{ \nu - \frac{d\pi}{dx} + \cdots \right\} \omega dx$$

$$+ h \int \left\{ \mathrm{N} - \frac{d\mathrm{P}}{dx} + \cdots \right\} \omega dx$$

$$- \int \left\{ h\mathrm{N} - \frac{d.h\mathrm{P}}{dx} + \cdots \right\} \omega dx;$$ where S is the sum of the definite parts: and correcting from $x = 0$ to $x = a$, there results

$$\delta\int_0^a \mathrm{V}dx = \mathrm{V}\delta x + \mathrm{S} + \int_0^a \left\{ \nu - \frac{d\pi}{dx} + \cdots \right\} \omega dx$$
$$+ \int_0^a \left\{ \mathrm{H_N} - \frac{d.\mathrm{H_P}}{dx} + \cdots \right\} \omega dx$$
$$- \int_0^a \left\{ h_\mathrm{N} - \frac{d.h_\mathrm{P}}{dx} + \cdots \right\} \omega dx$$
$$= \mathrm{V}\delta x + \mathrm{S} + \int_0^a \left\{ \nu - \frac{d\pi}{dx} + \cdots \right\} \omega dx$$
$$+ \int_0^a \left\{ k_\mathrm{N} - \frac{d.k_\mathrm{P}}{dx} + \frac{d^2.k_\mathrm{Q}}{dx^2} - \cdots \right\} \omega dx;$$

where $k = \mathrm{H} - h = \int_0^a \mathrm{T}dx - \int \mathrm{T}dx$.

In the case of a maximum or minimum, we have

$$\nu - \frac{d\pi}{dx} + \cdots + k_\mathrm{N} - \frac{d.k_\mathrm{P}}{dx} + \cdots = 0.$$

Ex. Let $d\mathrm{V} = \mu dx + \nu dy + \mathrm{T}ds$.

Here $dt = ds = \sqrt{1 + p^2}dx \therefore \mathrm{Z} = \sqrt{1 + p}$ $\therefore$

$d\mathrm{Z} = \dfrac{pdp}{\sqrt{1+p^2}}$. But we assume

$d\mathrm{Z} = \mathrm{M}dx + \mathrm{N}dy + \mathrm{P}dp + \mathrm{Q}dq + \cdots = 0 \therefore$

$\mathrm{N} = 0$, $\mathrm{P} = \dfrac{p}{\sqrt{1+p^2}}$, $\mathrm{Q} = 0$, $\cdots$ and the curve's equation

is $\nu - \dfrac{1}{dx} d \dfrac{kp}{\sqrt{1+p^2}} = 0$, or $\nu dx - d\dfrac{kp}{\sqrt{1+p^2}} = 0$.

28. *Required to find the variation of* $\int \mathrm{V}$ *when* V *is given by a fluxional equation of the first order.*

Let the given equation be $d\mathrm{V} + \mathrm{U}dx = \mathrm{X}dx$, where $\mathrm{U} = \mathrm{F}(\mathrm{V}, x, y, p, q \cdots)$ and $\mathrm{X} = f(x, y, p, q \cdots)$.

Taking the variation on the supposition that dx is constant, we have

$$d\delta\mathrm{V} + \left\{ \frac{d\mathrm{U}}{dx}\delta x + \frac{d\mathrm{U}}{dy}\delta y \right\} dx + \frac{d\mathrm{U}}{d\mathrm{V}}\mathrm{V}\,dx$$
$$- \left\{ \frac{d\mathrm{X}}{dx}\delta x + \frac{d\mathrm{X}}{dy}\delta y \right\} dx = 0.$$

Substituting $-\phi$ for the sum of the brackets, and T for $\dfrac{d\mathrm{U}}{d\mathrm{V}}$, there results $d\delta\mathrm{V} - \phi dx + \mathrm{T}\delta\mathrm{V}dx = 0$ (a).

To integrate this, multiply by the indeterminate quantity λ; then $\lambda d\delta v - \lambda\phi dx + \lambda T\delta v dx = 0$, or
$d.\lambda\delta v - d\lambda\delta v - \lambda\phi dx + \lambda T\delta v dx = 0$ (b).

Assume the sum of the second and fourth terms to $= 0$; then we have $d\lambda - \lambda T dx = 0$, or $\frac{d\lambda}{\lambda} = T dx$; and integrating, $\lambda = e^{\int T dx}$.

Substituting this in the sum of the first and third terms which $= 0$, and integrating, we have $e^{\int T dx}\delta v - \int e^{\int T dx}\phi dx = 0$; and consequently $\int \delta v$ or $\delta \int v = \int e^{-\int T dx} \int e^{\int T dx}\phi dx$.

29. *Required to find the variation of* $\int v$ *when* v *is given by a fluxional equation of a higher order than the first.*

Let the equation be $d^2v + u dx^2 = x dx^2$, where $u = F(v, x, y, v', p, q \cdots)$, denoting $\frac{dv}{dx}$ by v'.

Taking the variation as before, we shall have in equation (a) an additional term, viz. $\frac{du}{dv'}\delta v' dx = \frac{du}{dv'} d\delta v$. Substitute $T' = \frac{du}{dv'}$; then the additional term is $T' d\delta v$; and multiplying by λ as before, the new term in equation (b) is $\lambda T' d\delta v$, which $= d.\lambda T'\delta v - d.\lambda T'.\delta v$.

Hence, mak ng the sum of the second, fourth and sixth terms $= 0$, we shall have two equations:
$d\lambda - \lambda T dx + d.\lambda T' = 0$
$d.(\lambda + \lambda T')\delta v - \lambda\phi dx = 0$.

Find λ from the first of these equations, and substitute its value in the second; then, by integration, there will result the value of $\int \delta v$ or $\delta \int v$.

The same process applies if u also contains d^2v; and by repeating it, the variation may be found, when the given equation is of the third or of a higher order.

30. EXAMPLES.

Ex. 1. Required to draw the shortest line.

Since $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + p^2} dx \therefore \int \sqrt{1 + p^2} dx$ = minimum.

Here $v = \sqrt{1 + p^2} \therefore N = 0$; $P = \frac{p}{\sqrt{1 + p^2}}$;

$\mathrm{Q} = 0$; $\cdots$ and the equation $\mathrm{N} - \frac{d\mathrm{P}}{dx} + \cdots = 0$ gives

$\frac{d\mathrm{P}}{dx} = 0$, or $\mathrm{P} = \text{const.}$ $\therefore \frac{p}{\sqrt{1+p^2}} = \text{const.}$ $\therefore$ $p = c$, or $dy = cdx$ and $y = cx + c'$, a right line.

(1). Between two given points.

Here, eliminating c and c', the equation becomes

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

(2.) Between two parallel lines.

Let the lines be parallel to the axis Y; then, since $\delta x_I = 0$, the first equation of limits is $\mathrm{P}_I \delta y_I = 0$, or $\mathrm{P}_I = 0$ $\therefore$ $p_I = 0$, for $\mathrm{P} = \frac{p}{\sqrt{1+p^2}}$; or the required line is parallel to the axis X.

We are at liberty to divide the whole equation of limits $\mathrm{U}_{II} - \mathrm{U}_I = 0$ into two, because the limits are independent of each other (art. 22).

(3). Between two given curves.

The first equation of limits is $\mathrm{V}_I \delta x_I + \mathrm{P}_I \omega_I = 0$, or $(\mathrm{V}_I - \mathrm{P}_I p_I)\delta x_I + \mathrm{P}_I \delta y_I = 0$, or

$$\left\{ \sqrt{1 + p_I^2} - \frac{p_I^2}{\sqrt{1+p_I^2}} \right\} \delta x_I + \frac{p_I}{\sqrt{1+p_I^2}} \delta y_I = 0, \text{ or}$$

$\delta x_I + p_I \delta y_I = 0$.

From this we obtain $p_I = -\frac{\delta x_I}{\delta y_I}$. But if x_1, y_1 are the co-ordinates of the *curve* at the first limit, $\frac{\delta x_I}{\delta y_I} = \frac{dx_1}{dy_1}$, because the increments of the line's co-ordinates arising from any change in the position of the line must be the same as the increments of the curve's co-ordinates; whence we have $p_I = -\frac{1}{p_1}$; or $p_I p_1 + 1 = 0$. Similarly, $p_{II} p_2 + 1 = 0$; or the required line is at right angles to both the curves.

By means of these two equations combined with $y = cx + c'$ and the equations of the curves, the position of the required line may be determined.

Ex. 2. Required to draw the shortest line in space

Here $ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + p^2 + p'^2}dx \therefore$ $\mathrm{V} = \sqrt{1 + p^2 + p'^2}$.

Here $\mathrm{M} = 0$; $\mathrm{N} = 0 = \mathrm{N}'$; $\mathrm{P} = \dfrac{p}{\sqrt{1+p^2+p'^2}}$;

$\mathrm{P}' = -\dfrac{p'}{\sqrt{1+p^2+p'^2}}$; and the whole equation between the limits is $\dfrac{d\mathrm{P}}{dx}\delta y + \dfrac{d\mathrm{P}'}{dx}\delta z = 0$; which divides into $d\mathrm{P} = 0$ and $d\mathrm{P}' = 0$ by considering z and y separately constant (art. 24). Integrating, we have $\mathrm{P} = \text{const.}$ and $\mathrm{P}' = \text{const.}$ or

$\dfrac{p}{\sqrt{1+p^2}} = \text{const.}$ and $-\dfrac{p'}{\sqrt{1+p^2}} = \text{const.}$ or $p = \text{const.} = a$ and $p' = \text{const.} = b \therefore$ the required line is

$$\left.\begin{aligned} y &= ax + \alpha \\ z &= bx + \beta \end{aligned}\right\};$$ which is a right line in space.

(1). Between two given lines in space.

The first equation of limits is $\mathrm{V}_l\delta x_l + \mathrm{P}_l\omega_l + \mathrm{P}'_l\omega'_l = 0$, or $(\mathrm{V}_l - p_l\mathrm{P}_l - p'_l\mathrm{P}'_l)\delta x_l + \mathrm{P}_l\delta y_l + \mathrm{P}'_l\delta z_l = 0$; which by substitution becomes $\delta x_l + p_l\delta y_l + p'_l\delta z_l = 0$.

Since we may establish any relation ad libitum between δy and δz, this divides into $\delta x_l + p_l\delta y_l = 0$ and $\delta x_l + p'_l\delta z_l = 0$, or $p_l = -\dfrac{\delta x_l}{\delta y_l}$ and $p'_l = -\dfrac{\delta x_l}{\delta z_l}$.

Let the first limit be the right line $\left.\begin{aligned} y &= mx + \mu \\ z &= nx + \nu \end{aligned}\right\}$; and let x_1, y_1, z_1 be its co-ordinates at the limit; then

$m = \dfrac{dy_1}{dx_1} = \dfrac{\delta y_l}{\delta x_l} = -\dfrac{1}{p_l} = -\dfrac{1}{a}$. Similarly, $n = -\dfrac{1}{b}$, and consequently $am + bn + 1 = 0$; which is the condition that the line may cut the limit at right angles. (Vol. i. ch. 7. 67.) The same being true of the second limit, the required line is at right angles to both the given lines.

(2). Between two given surfaces.

Let x_1, y_1, z_1 be the co-ordinates of the surface at the first limit; and $x_1 = my_1 + nz_1 + c$ be the tangent plane at that point.

The equation of limits is the same as before. Also

$m = \frac{dx_1}{dy_1} = \frac{\delta x_1}{\delta y_1} = -a$. Similarly, $n = -b$; or the tangent plane is $x_1 + ay_1 + bz_1 = c$; which is a plane to which $\left.\begin{matrix} y = ax + \alpha \\ z = bx + \beta \end{matrix}\right\}$ is a normal (vol. i. ch. 7. 65.); which shows that the required line is at right angles to both the surfaces.

Ex. 3. Required to draw the shortest line upon a given surface, joining two given points in it.

Let $u = 0$ be the equation of the surface; and, when differentiated, let it give $dz = mdx + ndy$; then, as in the last example, we have $dP\delta y + dP'\delta z = 0$; but here y and z are connected by an equation of condition; wherefore (art. 25.), since δx may be supposed $= 0$, $\frac{du}{dy}\delta y + \frac{du}{dz}\delta z = 0$; and by elimination $\frac{du}{dz}dP - \frac{du}{dy}dP' = 0$; which, since

$P = \frac{p}{\sqrt{1+p^2+p'^2}} = \frac{dy}{ds}$, $P' = \frac{dz}{ds}$, $\frac{du}{dz} = 1$, $\frac{du}{dy} = -n$, becomes $d\frac{dy}{ds} + nd\frac{dz}{ds} = 0$ (1); which therefore belongs to the required line.

Since x and y may be interchanged in the equation $dz = mdx + ndy$, it is manifest that

$d\frac{dx}{ds} + md\frac{dz}{ds} = 0$ (2) also belongs to the required line.

Let the given surface be a sphere. Take the centre for the origin; then $x^2 + y^2 + z^2 = r^2$ ∴

$dz = -\frac{x}{z}dx - \frac{y}{z}dy$, or $m = -\frac{x}{z}$, $n = -\frac{y}{z}$ and (1) and (2) become $d\frac{dy}{ds} - \frac{y}{z}d\frac{dz}{ds} = 0$ and

$d\frac{dx}{ds} - \frac{x}{z}d\frac{dz}{ds} = 0$; or, if ds be constant,

$\frac{zd^2y - yd^2z}{ds} = 0$ and $\frac{zd^2x - xd^2z}{ds} = 0$; ∴ integrating,

$zdy - ydz = ads$ and $zdx - xdz = bds$ ∴

$zdy - ydz = c(zdx - xdz)$, whose requisite factor is $\frac{1}{z^2}$; ∴ integrating, $\frac{y}{z} = \frac{cx}{z} + c'$, or $y = cx + c'z$, which is a plane passing through the sphere's centre; or the required line is a great circle of the sphere.

Ex. 4. Required the curve joining two given points, which within its own arc, its evolute and radius of curvature shall contain the least area.

The radius of curvature $= \frac{(1+p^2)^{\frac{3}{2}}}{-q}$ (vol. i. ch. 11. 4.) ∴

$d.\text{area} = \frac{(1+p^2)^{\frac{3}{2}}ds}{-2q} = \frac{(1+p^2)^2dx}{-2q}$, or $v = \frac{(1+p^2)^2}{q}$.

From the form $v = aq + cp + c'$ (art. 23, form (2)), we have $\frac{(1+p^2)^2}{q} = -\frac{(1+p^2)^2}{q} + cp + c'$, or

$2(1+p^2)^2dx = (cp + c')dp$; or $2dx = \frac{(cp+c')dp}{(1+p^2)^2}$; ∴ integrating as in vol. i. ch. 2. 58. ex. 5,

$$4(x+a) = -\frac{c}{1+p^2} + c'\left\{\frac{p}{1+p^2} + \tan^{-1}p\right\} \quad . \quad . \quad . \quad . \quad .$$

$$= \frac{c'p-c}{1+p^2} + c'\tan.^{-1}p.$$

Also, $dy = pdx = \frac{(cp+c')pdp}{2(1+p^2)^2}$ . . . . . .

$$= \frac{cdp}{2(1+p^2)} - \frac{cdp}{2(1+p^2)^2} + \frac{c'pdp}{2(1+p^2)^2}$$ ∴ integrating,

$$2(y+b) = c\tan.^{-1}p - \frac{c}{2}\left\{\frac{p}{1+p^2} + \tan.^{-1}p\right\} - \frac{c'}{2}\frac{1}{1+p^2}, \text{ or}$$

$$4(y+b) = c\tan.^{-1}p - \frac{cp+c'}{1+p^2}.$$

Eliminating $\tan.^{-1}p$, the resulting equation is

$$4c(x+a) - 4c'(y+b) = \frac{2cc'p - c^2 + c'^2}{1+p^2}.$$

Since a and b are merely parameters of position, transfer the origin to the point (a, b), which will in no wise affect the curve, and the equation becomes

$$4cx - 4c'y = \frac{2cc'p - c^2 + c'^2}{1+p^2}.$$

The curve is a cycloid; for if, in $2dx = \frac{(cp+c')dp}{(1+p^2)^2}$, we suppose $c' = 0$, we have $2dx = \frac{cpdp}{(1+p^2)^2}$ $\therefore 4x = \frac{c}{1+p^2}$, if c be supposed negative, and consequently ds, which

$= \sqrt{1+p^2}dx$, $= \frac{c^{\frac{1}{2}}dx}{2x^{\frac{1}{2}}}$ $\therefore s = c^{\frac{1}{2}}x^{\frac{1}{2}}$, which is a cycloid.

There is the same result, if we suppose $c' = 0$.

The constants c and c' enter into the limits. If they are fixed points, and the position of the tangents of the curve at those points be given, *i. e.* p_I and p_{II}, we shall have the two equations $0 = \frac{2cc'p_I - c^2 + c'^2}{1+p_I^2}$ and $4cx_{II} = \frac{2cc'p_{II} - c^2 + c'^2}{1+p_{II}^2}$, by means of which c and c' are determined such that the cycloid shall touch the two given lines.

If the position of the tangents be not given, and the problem be, 'Of all cycloids that can be drawn through the two points, to find that which satisfies the proposed conditions,' we proceed thus.

The definite part of the whole variation

$$= \mathrm{V}\delta x + \left\{ \mathrm{P} - \frac{d\mathrm{Q}}{dx} \right\} \omega + \mathrm{Q}d\omega$$

$$= \mathrm{V}\delta x + \left\{ \mathrm{P} - \frac{d\mathrm{Q}}{dx} \right\} (\delta y - p\delta x) + \mathrm{Q}(\delta p - q\delta x)$$

$$= \left(\mathrm{V} - \mathrm{P}p - \mathrm{Q}q + \frac{pd\mathrm{Q}}{dx} \right)\delta x + \left\{ \mathrm{P} - \frac{d\mathrm{Q}}{dx} \right\} \delta y + \mathrm{Q}\delta p.$$

But, since the limits are fixed, δx_I, δy_I and δx_{II}, $\delta y_{II} = 0$; whence the whole equation of limits is $\mathrm{Q}_{II}\delta p_{II} - \mathrm{Q}_I \delta p_I = 0$.

Let θ_I and θ_{II} be the angles at which the curve is inclined to the axis at the limits when the conditions are satisfied; then, since $p_I = \tan.^{-1}\theta$, and $p_{II} = \tan.^{-1}\theta_{II}$, we have

$\delta p_I = (1 + p_I^2)\delta\theta_I$ and $\delta p_{II} = (1 + p_{II}^2)\delta\theta_{II}$; or the equation of limits becomes $\mathrm{Q}_{II}(1 + p_{II}^2)\delta\theta_{II} - \mathrm{Q}_I(1 + p_I^2)\delta\theta_I = 0$.

This, on account of the independence of θ_I and θ_{II}, divides into $\mathrm{Q}_I(1 + p_I^2) = 0$ and $\mathrm{Q}_{II}(1 + p_{II}^2) = 0$; but

$\mathrm{Q} = -\dfrac{(1+p^2)^2}{q^2}$, whence $\left(\dfrac{(1+p_I^2)^{\frac{3}{2}}}{q_I}\right)^2 = 0$ and

$\left(\dfrac{(1+p_{II}^2)^{\frac{3}{2}}}{q_{II}}\right)^2 = 0$; or the radii of curvature at the limits each $= 0$, and consequently the limits are cusps, and the required curve is a complete cycloid.

Ex. 5. Required the curve in which $\int\dfrac{d^2y^2}{dx^3} =$ maximum.

Here $\int\dfrac{d^2y^2}{dx^4}dx =$ maximum; $\therefore$ v $= q^2$; and since all the fluxional coefficients except $\mathrm{Q} = 0$, we have $\dfrac{d^2\mathrm{Q}}{dx^2} = 0$, or $d^2\mathrm{Q} = 0$; but $\mathrm{Q} = 2q \therefore d^2q = 0 \therefore dq = cdx \therefore q$ or $\dfrac{dp}{dx} = cx + c' \therefore p = \dfrac{cx^2}{2} + c'x + c'' \therefore$

$y = \dfrac{cx^3}{2.3} + \dfrac{c'x^2}{1.2} + c''x + c'''$.

The values of the arbitrary constants are to be determined from the conditions; thus, to take a particular case, suppose that two points through which the curve is to pass are given, viz. $(x = 0, y = 0)$ and $(x = a, y = b)$, and also the angles at which the curve at these points is inclined to the axis. Let t and t' be the tangents of these angles; then $(x = 0, y = 0)$ gives $c''' = 0$. Also, $x = 0$, $p = t$, give $t = c''$; $(x = a, y = b)$ gives $b = \dfrac{ca^3}{2.3} + \dfrac{c'a^2}{1.2} + ta$ and $x = a$, $p = t'$ give $t' = \dfrac{ca^2}{2} + c'a + t$; from which two equations, c and c' may be found.

Ex. 6. Required the spiral in which $\int\mathrm{R}ds =$ maximum, where R is a function of the radius vector and $s =$ the length of the arc.

Here $\mathrm{R}ds = \mathrm{R}\sqrt{dr^2 + r^2d\theta^2} = \mathrm{R}\sqrt{1 + r^2p^2}dr$, if

$p = \frac{d\theta}{dr} \therefore \text{N} = 0$ and $\text{P} = \frac{\text{R}r^2p}{\sqrt{1+r^2p^2}}$; but $d\text{P} = 0 \therefore$ $\frac{\text{R}r^2p}{\sqrt{1+r^2p^2}} = a$, or $d\theta = \frac{adr}{r\sqrt{\text{R}^2r^2-a^2}}$, which is the required equation.

Ex. 7. Required the curve in which $\int \frac{d^2y}{dxdy}$ = minimum.

Here $\text{v} = \frac{r}{p} \therefore$ (art. 23, form. 4) . . . . . .

$\frac{r}{p} = c' + cp + \frac{r}{p} - \frac{1}{dx}qd\frac{1}{p} \therefore 0 = c' + cp + \frac{q^2}{p^2}$; and taking one of the constants as negative that the result may not contain $\sqrt{-1}$, q or $\frac{dp}{dx} = p\sqrt{c' - cp}$, or

$dx = \frac{dp}{p\sqrt{c'-cp}}$; which may be reduced to a logarithmick form by substituting $z^2 = c' - cp$; and there results $e^{\frac{x\sqrt{c'}}{2}} = \frac{\sqrt{c'} - \sqrt{c'-cp}}{cp}$.

Ex. 8. Required the line in space in which $\int \frac{(y+z)dydz}{dx}$ = maximum or minimum.

Here $\text{v} = (y + z)pp' \therefore \text{M} = 0$; $\text{N} = pp' = \text{N}'$; $\text{P} = (y + z)p'$; $\text{P}' = (y + z)p$. Also s, which $= \text{v} - (\text{P}p + \text{P}'p')$, $= -(y + z)pp'$.

The three equations are $0 = -d.(y + z)pp'$, $pp'dx = d.(y + z)p' = d.(y + z)p$. (Art. 24. cor.)

From the first, $y + z = \frac{a}{pp'} \therefore$

$pp'dx = a.d.\frac{1}{p} = a.d.\frac{1}{p'} \therefore \frac{1}{p} = \frac{1}{p'} + b \therefore p' = \frac{p}{1-bp} \therefore$

$\frac{p^2dx}{1-bp} = a.d\frac{1}{p}$, or $dx = -\frac{a(1-bp)dp}{p^4} \therefore$

$x = \frac{a}{3p^3} - \frac{ab}{2p^2} + c.$

Also, $dy = pdx = -\frac{a(1-bp)dp}{p^3} \therefore y = \frac{a}{2p^2} - \frac{ab}{p} + c'$.

To find z in terms of p, we have

$y + z = \frac{a}{pp'} = \frac{a(1-bp)}{p^2}$; $\therefore z = \frac{a}{2p^2} - c'$.

From these we can find either of the three equations in space; thus, to find that between y and z, we have

$y = \frac{a}{2p^2} - \frac{ab}{p} + c' = z + 2c' - \frac{ab}{p} \therefore p = \frac{ab}{z-y+2c'}$ and the required equation is $2ab^2(z + c') = (z - y + 2c')^2$.

Ex. 9. Required the line in space in which $\int \sqrt{x^2 + y^2 + z^2}\sqrt{dx^2 + dy^2 + dz^2}$ = a maximum or minimum.

For abridgment, substitute $r = \sqrt{x^2 + y^2 + z^2}$, $s = \sqrt{1 + p^2 + p'^2}$; then $\mathrm{v} = rs$, $\mathrm{M} = \frac{sx}{r}$, $\mathrm{N} = \frac{sy}{r}$, $\mathrm{N}' = \frac{sz}{r}$, $\mathrm{P} = \frac{rp}{s}$, $\mathrm{P}' = \frac{rp'}{s}$. Also s, which

$= \mathrm{v} - (\mathrm{P}p + \mathrm{P}'p') = rs - \frac{rp^2+rp'^2}{s} = \frac{r}{s}$; wherefore the three equations are $\frac{sydx}{r} = d.\frac{rp}{s}(1)$;

$\frac{szdx}{r} = d.\frac{rp'}{s}(2)$; and $\frac{sxdx}{r} = d.\frac{r}{s}$ (3).

From (3) we have $xdx = \frac{r}{s}\, d.\frac{r}{s} \therefore$ integrating,

$x^2 + a^2 = \frac{r^2}{s^2}$; from which to obtain the equation between x and y, suppose z constant, or $s = \sqrt{1 + p^2} \therefore$

$x^2 + a^2 = \frac{r^2}{1+p^2} = \frac{x^2+y^2+b^2}{1+p^2}$, if $z = b$; or

$p^2(x^2 + a^2) = y^2 + b^2 - a^2 = y^2 + c^2$, if $c^2 = b^2 - a^2$;

$\therefore \frac{dy}{\sqrt{y^2+c^2}} - \frac{dx}{\sqrt{x^2+a^2}} = 0 \therefore$ integrating,

$\frac{y+\sqrt{y^2+c^2}}{x+\sqrt{x^2+a^2}} = n \therefore y - nx = n\sqrt{x^2 + a^2} - \sqrt{y^2 + c^2} \therefore$

$2n\sqrt{x^2+a^2}\sqrt{y^2+c^2} = n^2a^2 + c^2 + 2nxy$ ∴
$4n^2(c^2x^2 + a^2y^2) = (n^2a^2 - c^2)^2 + 4n(n^2a^2 + c^2)xy$, which belongs to a conick section. (Vol. i. ch. 7. 18. ex. 3.)

Similarly it may be shown that the projections on the planes XZ and YZ are also conick sections.

The curve lies wholly in one plane. Of this property Euler has given the following demonstration. (Mémoires de l'Acad. de Petersbourg, tom. iv. p. 3.)

From (1) and (3) $\frac{sydx}{r} = d.\frac{rp}{s} = pd.\frac{r}{s} + \frac{r}{s}dp$ . . .

$= \frac{dy}{dx}\frac{sxdx}{r} + \frac{r}{s}dp = \frac{sxdy}{r} + \frac{r}{s}dp$ ∴ $ydx - xdy = \frac{r^2}{s^2}dp$.

Similarly, $zdx - xdz = \frac{r^2}{s^2}dp'$ ∴ $\frac{dp}{dp'} = \frac{ydx-xdy}{zdx-xdz} = \frac{y-xp}{z-xp}$

∴ $\frac{dp}{y-xp} = \frac{dp'}{z-xp'}$ ∴ integrating, $\frac{y-xp}{z-xp} = n$, which ∴

$= \frac{dp}{dp'}$ ∴ $p = np' + a$, or $dy = ndz + adx$ ∴ $y = nz + ax + b$,

which belongs to a plane.

Euler in his next example investigates the nature of the curve in which $\int \text{R}sdx$ is a maximum, R being any function of r. It is manifest that this curve lies wholly in one plane; and if we transfer the rectangular into polar co-ordinates, r becomes the radius vector; and it may be solved as in ex. 6.

Ex. 10. The brachystochrone.

The velocity at any point of the descent is as $\sqrt{y}$ ∴ the time $= \int \frac{ds}{\sqrt{y}} = \int \frac{\sqrt{1+p^2}dx}{\sqrt{y}}$, or $\text{v} = \frac{\sqrt{1+p^2}}{\sqrt{y}}$; which contains only y and p, and consequently $\text{v} = \text{P}p + \frac{1}{c}$, or

$\frac{\sqrt{1+p^2}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\sqrt{1+p^2}} + \frac{1}{c}$ ∴ $\frac{1}{\sqrt{y}\sqrt{1+p^2}} = \frac{1}{c}$, or

$\frac{ds}{dx} = \frac{c^{\frac{1}{2}}}{y^{\frac{1}{2}}}$; which belongs to a cycloid, the diameter of whose generating circle $= c$.

(1). Between two given points.

Take the higher point for the origin of the co-ordinates; then, since $\frac{1}{\sqrt{y}\sqrt{1+p^2}} = \frac{1}{\sqrt{c}}$, we have $1 + p^2 = \frac{c}{y}$, or

$p = \frac{\sqrt{c-y}}{\sqrt{y}}$ $\therefore dx = \frac{y^{\frac{1}{2}}dy}{\sqrt{c-y}} = \frac{ydy}{\sqrt{cy-y^2}}$ $\therefore$ integrating,

$x = \frac{c}{2}v.s^{-1}\frac{2y}{c} - \sqrt{cy-y^2}$, for $c' = 0$: which is an inverted cycloid whose base is horizontal.

The points being given, this determines the cycloid *.

At the origin where $x = 0$, $y = 0$, we have $p = \infty$; or the cusp of the cycloid is at the origin.

It is here supposed that the body begins to descend from rest: if it have any initial velocity, the curve is a cycloid as before, and may be determined from the data.

(2). Between two given curves, the velocity being that acquired in falling from a given horizontal line.

Taking the origin in the given horizontal line, we have, as before, $v - vp = \frac{1}{c}$ and $\sqrt{y}\sqrt{1+p^2} = c$ $\therefore$

$v = \frac{p}{\sqrt{y}\sqrt{1+p^2}} = \frac{p}{c}$.

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* This is the problem which was proposed by John Bernoulli as one of great difficulty and importance in the Leipsic Acts. It was sent to Newton, and on the following day (30 Jan. $169\frac{6}{7}$) he gave the solution which is inserted with that of another problem, which Bernoulli proposed at the same time, in the Philosophical Transactions, No. 224, p. 384. It was evident from the Principia, that Newton possessed the means of solving problems of this class. He did not put his name to the solution, upon which Bernoulli observes: 'Quoique l'auteur de cette construction, par un excès de modestie, ne se nomme pas, nous savons pourtant indubitablement par plusieurs circonstances que c'est le célèbre Mr. Newton: et quand même nous ne le saurions point d'ailleurs, ce seroit assez de le connoître par cet échantillon comme *ex ungue leonem.*' —Joh. Bernoulli, tom. i. p. 196.

The problems of the next chapter were first proposed by James Bernoulli. For the history of this branch of the fluxional calculus, which is peculiarly interesting, the reader should consult the late Prof. Woodhouse's Treatise on Isoperimetrical Problems.

The indefinite part of the whole variation $= v\delta x + p\omega = (v - pp)\delta x + p\delta y = \frac{\delta x}{c} + \frac{p\delta y}{c}$ ∴ the whole equation of limits is $\delta x_{II} - \delta x_I + p_{II}\delta y_{II} - p_I\delta y_I = 0$, which divides into $\delta x_I + p_I\delta y_I = 0$ and $\delta x_{II} + p_{II}\delta y_{II} = 0$. But, if p_1, p_2 be the values of $\frac{dy}{dx}$ belonging to the curves at the limits, we have $\frac{\delta y_I}{\delta x_I} = p_1$ and $\frac{\delta y_{II}}{\delta x_{II}} = p_2$; ∴ $1 + p_I p_1 = 0$ and $1 + p_{II}p_2 = 0$; which shows that the cycloid cuts both the curves at right angles.

The cusp of the cycloid is not in the first curve, for p_I is not $= \infty$, but in the given horizontal line.

Ex. 11. Let a body move uniformly along a curve from one given point to another, and press upon the curve with a force which varies inversely as the square of the distance from a fixed point; required the curve when the whole pressure is a minimum.

Consider the curve as a spiral; then the pressure at $p = \frac{1}{r^2}$; and since the velocity is uniform, $d.$ pressure $= \frac{ds}{r^2} = \frac{\sqrt{p^2 + r^2}}{r^2} d\vartheta$ where $p = \frac{dr}{d\vartheta}$.

Here $v = \frac{\sqrt{p^2 + r^2}}{r^2}$ ∴ from the form $v = pp + \frac{1}{c}$, we have $\frac{\sqrt{p^2 + r^2}}{r^2} = \frac{p^2}{r^2\sqrt{p^2 + r^2}} + \frac{1}{c}$, or $\sqrt{p^2 + r^2} = c$; which is the spiral equation of a circle which passes through the pole.

Cor. If the velocity is as r^n, $d.$ pressure $= \frac{ds}{r^{n+2}}$ and the equation is $r^n\sqrt{p^2 + r^2} = c$.

Ex. 12. Required the curve which, by its revolution, generates the solid of least resistance; the solid moving in the direction of the axis.

The resistance on the surface generated by ds estimated in the direction of the axis is as surface $ds \times \sin.^3 \angle$ which ds makes with the axis, is as $yds.\frac{dy^3}{ds^3}$ is as $\frac{ydy^3}{ds^2}$; ∴ the whole re-

sistance $= \left(\int \frac{y dy^3}{ds^2}\right) = \left(\int \frac{yp^3 dx}{1+p^2}\right) \therefore$

$v = \frac{yp^3}{1+p^2}$; $p = \frac{3yp^2}{1+p^2} - \frac{2yp^4}{(1+p^2)^2}$ $\therefore$ the required curve is $\frac{yp^3}{1+p^2} = \frac{3yp^3+yp^5}{(1+p^2)^2} + c \therefore$ changing c, $yp^3 = c(1+p^2)^2$, or $\frac{ydy^3dx}{ds^4} = c$.

The curve does not meet the axis; for $\frac{y}{c} = \frac{ds^4}{dy^3dx}$, and since $ds^4 > dy^3dx$, we have $y > c$; or the required solid is *obtruncated*.

It can be easily shown that the angle which the curve makes with the least ordinate = the supplement of 45°; wherefore the fluxional triangle corresponding to the least ordinate is isosceles, or $ds^4 = 4dy^4 \therefore (y) = 4c$.

If the least ordinate be given, and be denoted by (y), we shall have $\frac{ydy^3dx}{ds^4} = \frac{(y)}{4}$, or $y = \frac{(y)^4 ds^4}{dy^4} \div \frac{4(y)dx}{dy}(y)^2$, which is the construction given, Princ. vol. ii. p. 34. Scholium *.

To find the curve which is a transcendental, we have $yp^3 = c(1+p^2)^2 \therefore y = c(p^{-3} + 2p^{-1} + p) \therefore$
$pdx = dy = -c(3p^{-4} + 2p^{-2} - 1)dp \therefore$
$dx = -c(3p^{-5} + 2p^{-3} - p^{-1})dp \therefore$

$x = c\left(\frac{3p^{-4}}{4} + p^{-2} + lp\right) + c'$; and combining this with $yp^3 = c(1+p^2)^2$, we shall have the relation between x and y.

If the axis of the solid and its greatest ordinate be given, c may be found from the resulting equation, and the curve may be determined.

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* The first edition of the Principia appeared in 1687.

CHAPTER II.

Isoperimetrical Problems.

1. By means of the three formulæ of arts. 17, 27 and 28 of the preceding chapter, we can solve all problems which involve only the maximum or minimum property. The first is to be used when v is a determinate function of $(x, y, z \cdots p, q \cdots p', q' \cdots)$; the second, when v includes an indeterminate fluent, such as s, the length of the curve, or even when v contains a double or a triple fluent; and the third, when v is given by a fluxional equation of the first or of a higher order.

There is a class of problems which, in addition to the maximum property, involve one or more of what are termed *isoperimetrical* properties. Thus the curve of swiftest descent from one point or curve to another, or the brachystochrone, belongs to the first class; but if it be required to find the brachystochrone when of a given length, the problem involves one isoperimetrical property; and if to this be added that it shall also contain a given area, it involves two isoperimetrical properties. We shall proceed, then, to adapt our formulæ to the case of isoperimetrical problems.

This reduction of relative to the case of absolute maxima or minima is accomplished by means of the following theorem.

2. *If at the same time that one function, as u, is to be a maximum or a minimum, other functions, as v, w, . . ., are given; then the required curve is the same as that which would result from* $\delta(u + av + bw + \cdots) = 0$, *where* $a, b \cdots$ *are constant quantities.*

For, suppose that there is only one isoperimetrical property, viz. that the function v is given.

Let δu, which $= 0$, $= A\delta x + B\delta y$; where, if there is no isoperimetrical property, δx and δy are wholly indeterminate.

Next, suppose δv, which $= 0$, $= A'\delta x + B'\delta y$; then δx and δy in $A\delta x + B\delta y = 0$ must be no longer indeterminate, but must also satisfy the new hypothesis or the equation $A'\delta x + B'\delta y = 0$.

Now (ex hyp.) $\delta(u + av)$, $= 0$, $= (A + aA')\delta x + (B + aB')\delta y$, in which δx and δy satisfy both $A\delta x + B\delta y = 0$ and $A'\delta x + B'\delta y = 0$ conjointly, and neither without the other; wherefore the proposed is reduced to the case of an absolute maximum or minimum, viz. $\delta(u + av) = 0$.

The introduction of the constant a enables us to give to v the value required by the conditions of the problem.

The same method is manifestly applicable, whatever be the number of isoperimetrical properties.

3. EXAMPLES.

Ex. 1. Required the curve of given length whose area shall be a maximum.

Here $u + av = \int(y + a\sqrt{1+p^2})dx \therefore v = y + a\sqrt{1+p^2}$; and from the form $v = Pp + c$, we have

$$y + a\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + c, \text{ or } \frac{a}{\sqrt{1+p^2}} = c - y \therefore$$

$$p = \frac{\sqrt{a^2-(c-y)^2}}{c-y} \therefore dx = \frac{(c-y)dy}{\sqrt{a^2-(c-y)^2}} \therefore$$

$x - c' = \sqrt{a^2 - (c-y)^2}$, or $(x - c')^2 + (y - c)^2 = a^2$; which belongs to a circular arc.

(1). If the limits be fixed, viz. (x', y') and (x'', y''), then we have

$$\left.\begin{aligned}(x' - c')^2 + (y' - c)^2 &= a^2\\ (x'' - c')^2 + (y'' - c)^2 &= a^2\end{aligned}\right\}, \text{ by means of which } c \text{ and } c'$$

may be each expressed in terms of the limits and of the constant a, the quantity upon which the length of the line depends.

The chord of the arc $= \sqrt{(x'' - x')^2 + (y'' - y')^2}$, and consequently we can find the radius a in order that the area included by y', y'' and the given arc may be a maximum; or a minimum, if the arc be drawn convex to the axis.

Cor. A curve of given length must be formed into a circular arc in order that it may include with a given right line the greatest area possible.

(2). Next, let only y' and y'', or y_I and y_{II} be given.

Here $\delta y_I = 0$ and $\delta y_{II} = 0$; and the whole equation of

limits becomes $(v_{II} - p_{II}p_{II})\delta x_{II} - (v_{,} - p_{,}p_{,})\delta x_{,} = 0$; which divides into $v_{II} - p_{II}p_{II} = 0$ and $v_{,} - p_{,}p_{,} = 0$.

But $v = pp + c$ or $v - rp = c$; wherefore in this case $c = 0$; and the curve's equation becomes $(x - c')^2 + y^2 = a^2$; which belongs to a circular arc whose centre is in the axis x; or the area intercepted by two moveable and given ordinates and an arc of given length is a maximum, when that arc is part of a circle whose centre is in the axis x.

(3). Lastly, let $y' = 0 = y''$.

Here $x_{II} - x_{,}$ is the chord of the arc; and since the centre is to be in $x_{II} - x_{,}$, it appears that the curve of given length, which with its chord includes the greatest area, is the semicircle.

Ex. 2. Required the curve which shall include a given area with the least perimeter.

Here $v = \sqrt{1 + p^2} + ay$; and from the form $v = pp + c$, we have $\sqrt{1 + p^2} + ay = \dfrac{p^2}{\sqrt{1+p^2}} + c$, or

$\dfrac{1}{\sqrt{1+p^2}} = c - ay$; which is a circle, the same as in ex. 1.

Cor. This commutability of the properties, first observed by James Bernoulli, evidently results from the article; for, a being constant, $\delta(u + av) = 0$ must give the same curve as $\delta(v + au) = 0$.

Ex. 3. The surface generated by a curve revolving round its axis being given; required the curve that the solid generated may be a maximum.

Here $v = y^2 + ay\sqrt{1 + p^2}$ $\therefore$ from the form $v = pp + c$, we have $y^2 + ay\sqrt{1 + p^2} = \dfrac{ayp^2}{\sqrt{1+p^2}} + c^2$, or

$\dfrac{ay}{\sqrt{1+p^2}} = c^2 - y^2$, or $dx = \dfrac{(c^2-y^2)dy}{\sqrt{a^2y^2-(c^2-y^2)^2}}$.

This, when integrated, will contain three arbitrary constants, which are to be determined from the data. Thus, if the limits are two fixed points, the constants must be determined so that the curve may pass through the two points, and that its surface may be a given quantity.

If the extreme ordinates of the curve be given in magnitude, but not in position; then, since $\delta y_{,}$ and δy_{II} each $= 0$,

the equation of limits becomes
$(v_{II} - p_{II}p_{II})\delta x_{II} - (v_I - p_I p_I)\delta x_I = 0$, or $c^2\delta x_{II} - c^2\delta x_I = 0$; which, on account of the independence of δx_{II} and δx_I, gives $c^2 = 0$; or the required curve is

$dx = \frac{-y^2 dy}{\sqrt{a^2y^2 - y^4}} = \frac{-ydy}{\sqrt{a^2 - y^2}}$; and integrating,

$x + c' = \sqrt{a^2 - y^2}$; which is a circle whose centre is in x. This, together with the isoperimetrical property, determines the position of the ordinates.

If $y' = 0 = y''$, the curve is a semicircle; and the solid, a whole sphere.

If we suppose that the circular ends may form part of the given surface, we shall obtain the same result.

For, if we include the circular ends whose radii are y_I and y_{II}, we shall have

$$\delta\left(\int (y^2 + ay\sqrt{1+p^2})dx + a\left\{\frac{y_I^2}{2} + \frac{y_{II}^2}{2}\right\}\right) = 0.$$

The indefinite part of this equation is the same as before; but the definite part is increased by $\frac{a}{2}\delta(y_I^2 + y_{II}^2)$, which $= a(y_I\delta y_I + y_{II}\delta y_{II})$; or the whole equation of limits is $v_{II}\delta x_{II} - v_I\delta x_I + p_{II}\omega_{II} - p_I\omega_I + a(y_{II}\delta y_{II} + y_I\delta y_I) = 0$, or $(v_{II} - p_{II}p_{II})\delta x_{II} - (v_I - p_I p_I)\delta x_I + (p_{II} + ay_{II})\delta y - (p_I - ay_I)\delta y_I = 0$.

But $v - pp = c^2$; for the equation of the curve is the same as before; also $p = \frac{ayp}{\sqrt{1+p^2}}$ ∴ the equation of limits is

$$c^2(\delta x_{II} - \delta x_I) + \left\{\frac{p_{II}}{\sqrt{1+p_{II}^2}} + 1\right\}ay_{II}\delta y_{II} \quad \ldots \ldots$$
$$- \left\{\frac{p_I}{\sqrt{1+p_I^2}} - 1\right\}ay_I\delta y_I = 0.$$

But δx_I, δx_{II} and δy_I, δy_{II} are wholly independent. Also, the points (x_I, y_I) and (x_{II}, y_{II}) not being constrained to move in any curve, δx_I, δy_I and δx_{II}, δy_{II} are wholly independent ∴ we have $c^2 = 0$; $\left\{\frac{p_I}{\sqrt{1+p_I^2}} - 1\right\}y_I = 0$ and $\left\{\frac{p_{II}}{\sqrt{1+p_{II}^2}} + 1\right\}y_{II} = 0$. These are satisfied either by $y_I = 0$, $y_{II} = 0$, or by $p_I = \infty$, $p_{II} = -\infty$; both of which

agree in giving the curve a semicircle, or the solid a whole sphere as before. (Airy's Math. Tracts, p. 178).

Ex. 4. Required the curve of given length and including with its axis a given area, which by revolving round the axis generates the greatest solid.

Here $u = \int y^2 dx$; $v = \int \sqrt{1+p^2}dx$ and $w = \int y dx$ $\therefore$ $u + av + bw = \int (y^2 + a\sqrt{1+p^2} + by)dx$, or $\mathrm{v} = y^2 + by + a\sqrt{1+p^2}$.

Since $\mathrm{v} = \mathrm{P}p + c$, we have

$$y^2 + by + a\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + c^2, \text{ or}$$

$$y^2 + by - c^2 = -\frac{a}{\sqrt{1+p^2}}, \text{ or } 1 + p^2 = \frac{a^2}{(y^2-by-c^2)^2} \therefore$$

$$p = \frac{\sqrt{a^2-(y^2-by-c^2)^2}}{y^2-by-c^2}, \text{ or } dx = \frac{(y^2-by-c^2)dy}{\sqrt{a^2-(y^2-by-c^2)^2}};$$

which belongs to the *elastick* curve.

When this is integrated, the primitive contains *four* arbitrary constants: two of these are determined by the assigned values of the length of the curve and of its area; the remaining two depend upon the limits.

Cor. If the isoperimetrical property of the given area be omitted, *i. e.* if $b = 0$, $dx = \frac{(y^2-c^2)dy}{\sqrt{a^2-(y^2-c^2)}}$.

Ex. 5. Required the curve in which $(\int y^m dx)$ is given, and $\left(\int \frac{y^r(dx^2 \pm dy^2)^n}{dy^{2n-1}}\right)$ is a maximum or minimum.

Here $u = \int \frac{y^r(1 \pm p^2)^n dx}{p^{2n-1}}$ and $v = \int y^m dx$ $\therefore$

$$\mathrm{v} = \frac{y^r(1 \pm p^2)^n}{p^{2n-1}} + ay^m.$$

Here $\mathrm{P} = y^r \left\{ \frac{\pm 2n(1 \pm p^2)^{n-1}}{p^{2n-2}} - \frac{(2n-1)(1 \pm p^2)^n}{p^{2n}} \right\}$ $\therefore$

from the form $\mathrm{v} = \mathrm{P}p + c$, we have

$$\frac{y^r(1 \pm p^2)^n}{p^{2n-1}} + ay^m = y^r \left\{ \frac{\pm 2n(1 \pm p^2)^{n-1}}{p^{2n-3}} - \frac{(2n-1)(1 \pm p^2)^n}{p^{2n-1}} \right\}$$

$+ c$, or $\frac{2ny^r(1 \pm p^2)^{n-1}}{p^{2n-1}} + ay^m = c$ (1).

If $c = 0$, there results $\frac{y^{r-m}(1+p^2)^{n-1}}{p^{2n-1}} = \text{const.}$; which is the fundamental theorem by which T. Simpson solves all problems *de maximis et minimis*. (Simpson's Fluxions, vol. ii. p. 206.) They are only particular solutions; but may be rendered general by the formula (1).

Ex. 6. Required the form of a curve of given length suspended from two fixed points, that its centre of gravity may be the lowest possible.

Let x be horizontal and y vertical; then the distance of the centre of gravity from $x = \frac{\int y ds}{s} = \frac{\int y ds}{b}$, if b = the given length.

Hence $\mathrm{v} = y\frac{\sqrt{1+p^2}}{b} + a\sqrt{1+p^2} \therefore \mathrm{P} = \frac{(y+ab)p}{b\sqrt{1+p^2}} \therefore$ from the form $\mathrm{v} = \mathrm{P}p + c$, we have

$$\frac{y+ab}{b}\sqrt{1+p^2} = \frac{(y+ab)p^2}{b\sqrt{1+p^2}} + c, \text{ or } \frac{y+ab}{b\sqrt{1+p^2}} = c.$$

To integrate this, we have $\frac{y+ab}{bc} = \sqrt{1+p^2}$, or

$$p = \frac{\sqrt{(y+ab)^2 - b^2c^2}}{bc} \therefore dx = \frac{bc\,dy}{\sqrt{(y+ab)^2 - b^2c^2}}, \text{ and integrating, } \frac{x}{bc} = l\frac{y+ab+\sqrt{(y+ab)^2-b^2c^2}}{c'};$$

which is the catenary.

If the isoperimetrical condition be omitted, $a = 0$; and the equation becomes $c = \frac{y}{b\sqrt{1+p^2}} = \frac{y\,dx}{b\,ds}$, or $\frac{ds}{dx} = \frac{y}{m}$, which is the same curve as before, m representing the tension at the lowest point.

If the limits be curves, it may be shown that the catenary will intersect them at right angles.

Ex. 7. Of all isochronous curves drawn from one given point to another, required that which includes with its chord the greatest area.

$$d.\,\text{time} = \frac{ds}{\text{vel.}} = \frac{\sqrt{1+p^2}\,dx}{\sqrt{y}} \therefore \left(\int \frac{\sqrt{1+p^2}}{\sqrt{y}}dx\right) = \text{a given quantity.}$$

Also $\left(\int ydx - \frac{xy}{2}\right)$ = maximum, or $\left(\int (ydx - d.\frac{xy}{2})\right) \cdots$ = maximum = $(\int (ydx - xdy)) = (\int (y - xp)dx)$.

Hence $v = y - xp + \frac{a\sqrt{1+p^2}}{\sqrt{y}}$ . . . . . .

$= -xp + \frac{ap^2}{\sqrt{y}\sqrt{1+p^2}} + c$, or $\frac{a}{\sqrt{y}\sqrt{1+p^2}} = c - y$; and by reduction, $dx = \frac{y^{\frac{1}{2}}(y-c)dy}{\sqrt{a^2-y(y-c)^2}}$.

If $c = 0$, the equation becomes $dx = \frac{y^{\frac{3}{2}}dy}{\sqrt{a^3-y^3}}$.

The isochronous curve is a cycloid; for to find it, we have

$\delta(\int \frac{\sqrt{1+p^2}}{\sqrt{y}}dx) = 0 \therefore \frac{\sqrt{1+p^2}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\sqrt{1+p^2}} + \frac{1}{\sqrt{c}}$, or

$\sqrt{y}\sqrt{1+p^2} = \sqrt{c} \therefore \frac{ds}{dx} = \sqrt{\frac{c}{y}}$.

Ex. 8. Required the solid of least resistance, when the area of the generating curve is given.

Here $v = \frac{yp^3}{1+p^2} + ay \therefore r = \frac{3yp^2+yp^4}{(1+p^2)^2}$; and the required curve is $\frac{yp^3}{1+p^2} + ay = \frac{3yp^3+yp^5}{(1+p^2)^2} + c \therefore$

$ay = \frac{2yp^3}{(1+p^2)^2} + c$.

This equation when integrated contains three arbitrary constants, and they are to be determined from the greatest ordinate, the length of the axis and the area of the generating plane.

If $c = 0$, we have $a = \frac{2p^3}{(1+p^2)^2} \therefore p = \text{const.}$; or the curve is a right line.

(2). When the length of the curve is given.

Here $v = \frac{yp^3}{1+p^2} + a\sqrt{1+p^2} \therefore r = \frac{3yp^2+yp^4}{(1+p^2)^2} + \frac{ap}{\sqrt{1+p^2}}$; and the required curve is

$\frac{yp^3}{1+p^2} + a\sqrt{1+p^2} = \frac{3yp^3+yp^5}{(1+p^2)^2} + \frac{ap^2}{\sqrt{1+p^2}} + c$, or

$\frac{a}{\sqrt{1+p^2}} = \frac{2yp^3}{(1+p^2)^2} + c.$

(3). When the solid content is given.

The required equation is $ay^2 = \frac{2yp^3}{(1+p^2)^2} + c.$

If $c = 0$, we have $ay = \frac{p^3}{(1+p^2)^2}$, which is integrated, ch. 5. 2. ex. 3.

(4). When the surface is given.

Here $v = \frac{yp^3}{1+p^2} + ay\sqrt{1+p^2}$ ∴

$p = \frac{3yp^2+yp^4}{(1+p^2)^2} + \frac{ayp}{\sqrt{1+p^2}}$ ∴ we have

$\frac{yp^3}{1+p^2} + ay\sqrt{1+p^2} = \frac{3yp^3+yp^5}{(1+p^2)^2} + \frac{ayp^2}{\sqrt{1+p^2}} + c$, or

$\frac{ay}{\sqrt{1+p^2}} = \frac{2yp^3}{(1+p^2)^2} + c.$

If $c = 0$, the curve is a right line.

(5). When both the solid content and the surface are given.

Here $v = \frac{yp^3}{1+p^2} + ay^2 + by\sqrt{1+p^2}$ ∴

$p = \frac{3yp^2+yp^4}{(1+p^2)^2} + \frac{byp}{\sqrt{1+p^2}}$, and we have

$\frac{yp^3}{1+p^2} + ay^2 + by\sqrt{1+p^2} = \frac{3yp^3+yp^5}{(1+p^2)^2} + \frac{byp^2}{\sqrt{1+p^2}} + c$, or

$ay^2 + \frac{by}{\sqrt{1+p^2}} = \frac{2yp^3}{(1+p^2)^2} + c.$

PRAXIS.

1. Required the curve in which $\int(ax - y^2)ydx =$ maximum. $y = \sqrt{\frac{ax}{3}}$.

2. Required the curve in which $\int \frac{d^2y}{ds} =$ maximum. A right line.

3. Required the curve in which $\int \frac{d^2y}{yds} =$ maximum $\therefore$ $cp^2 = y^2\sqrt{1+p^2}$.

4. Required the curve in which $\int (x^2+y^2)^n ds =$ maximum $\tan.\text{B} = \frac{\tan.2n\text{A} - c}{1 + c\tan.2n\text{A}}$; where $\text{A} = \tan.^{-1}\frac{x}{y}$ and $\text{B} = \tan.^{-1}p$.

5. Required the line in space in which $\int \frac{xdydz}{dx} =$ maximum or minimum. The line lies wholly in one plane, which is at right angles to YZ, and its projections on XY and XZ are logarithmick curves.

6. Of all curves of a given length, required that which by its revolution generates the greatest surface. $(2\pi y + a)ds = adx$; which is a catenary revolving round an horizontal axis; and the curve is concave to the axis when the surface is to be a minimum.

7. Required the brachystochrone, when the velocity of descent varies as the ordinate of the curve. A circle.

8. Required the brachystochrone, when the velocity varies as any function of the ordinate $\frac{ds}{dx} = \frac{a}{\text{Y}}$.

9. Required the brachystochrone, when the velocity varies as the nth power of the distance from a given point $r^{n-2}\sqrt{p^2+r^2} = c$.

   Cor. 1. If $n = 0$, or the velocity is uniform, the curve becomes a right line, where c is the perpendicular drawn from the pole.

   Cor. 2. If $n = 1$, the curve is the equiangular spiral.

   Cor. 3. If $n = 2$, the curve is a circle passing through the pole.

10. Given the axis and the greatest ordinate of two equal piers or starlings of the arch of a bridge; required the curve, that the resistance of the stream may be a minimum. $x = \frac{cp}{(1+p^2)^2}$ and $y = px + \frac{c}{2(1+p^2)}$. If

the span of the arch is infinite, or there is only one pier, the curve becomes a right line.

11. Given the extreme points and the length of a spiral arc; required the curve when the spiral area is a maximum. A circle whose radius $= \frac{1}{2}\sqrt{a^2 + c^2}$ and the distance of pole from centre $= \frac{1}{2}\sqrt{\frac{a^2}{2} - c^2}$.

12. A spiral area is included within three lines of given length; required the curve line when the area is a maximum. A circle which passes through the pole.

13. Required the brachystochrone from a given point to a given vertical line, when the velocity of descent is uniform and the whole area is given. A circle.

14. Required the brachystochrone from a given horizontal to a given vertical line, when the whole area is given.

If $c = 0$, $dx = \frac{y^{\frac{3}{2}}dy}{\sqrt{a^3 - y^3}}$; and if the velocity of descent is to be uniform, the curve is a circle.

15. Required the brachystochrone of given length between two given points

$$dy = \frac{cx^{\frac{1}{2}}dx}{\sqrt{4ac^2 + (b^2 - c^2)x \pm 4bca^{\frac{1}{2}}x^{\frac{1}{2}}}}.$$

16. Given the axis and the area of the generating curve of a solid of revolution; required its form, in order that it may be emptied through a small orifice at the vertex in the least time. $xy^2 = a^3$.

FINIS.

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